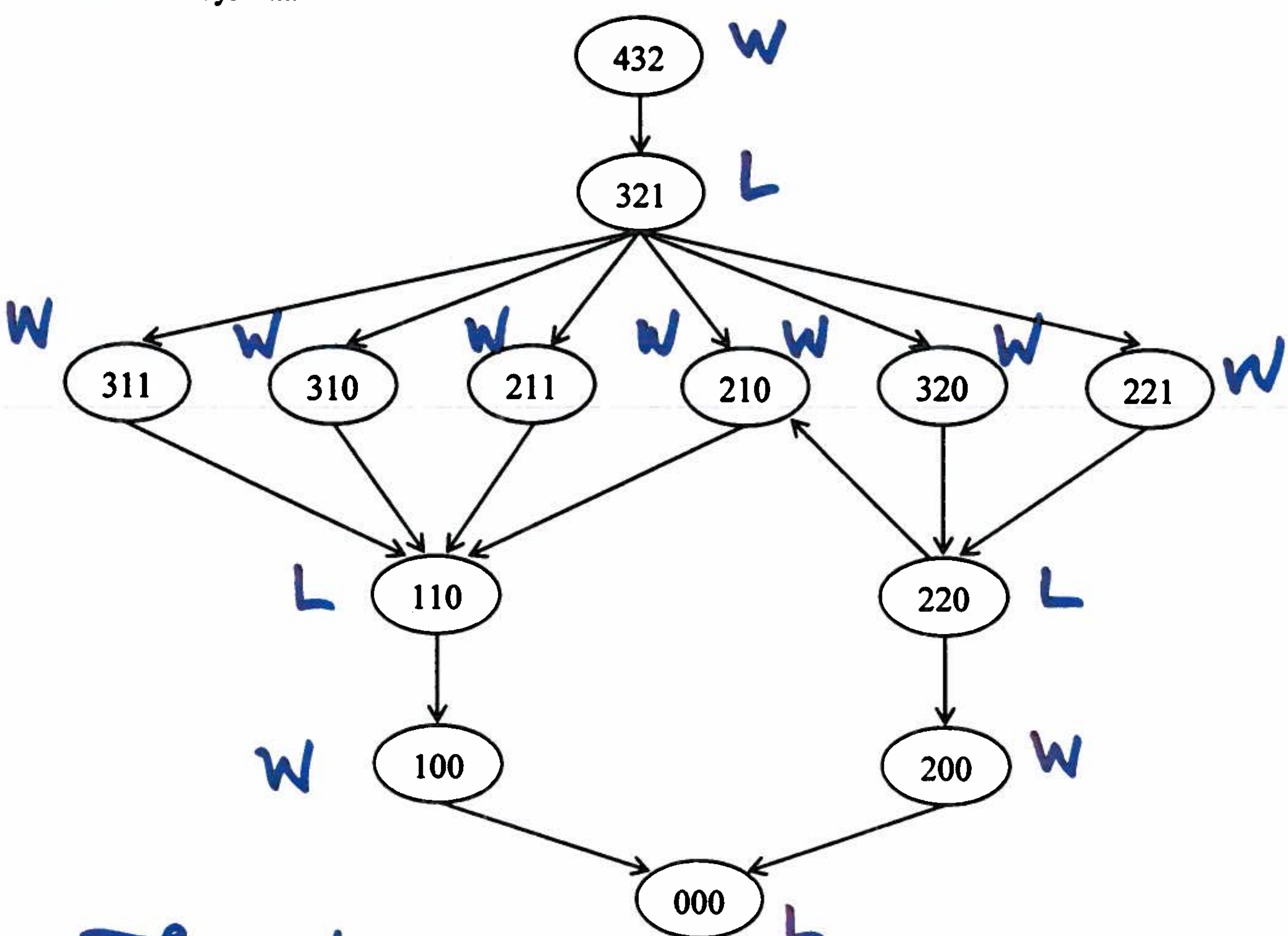
Lecture 5

Nim

How to play: Place 9 counters as shown. Players take turns removing counters, provided they are all in the same (horizontal) row. The player that picks up the last counter wins the game. State the player that can always win.

Represent a state of this game by three numbers. Each of the three numbers will correspond to one of the three rows. Each number will track how many counters remain in its corresponding row. By symmetry, write the larger numbers on the left. This means the game starts in the state 432. With this representation the following portion of the state diagram will reveal the player that can always win.



The 1st player is in a winning position initially.

Theorem: Every positive integer can be written uniquely as a sum of powers of 2, e.g. if $a > 0$ then

$$a = r_n \cdot 2^n + r_{n-1} \cdot 2^{n-1} + r_{n-2} \cdot 2^{n-2} + \cdots + r_1 \cdot 2 + r_0$$

where the coefficients are unique and $0 \le r_i \le 1$, so $r_i = 0$ or 1 for each i.

Proof: Divide a by 2 to get

$$a = 2 \cdot q_0 + r_0, \qquad r_0 = 0 \text{ or } 1$$

divide q_0 by 2 to get

$$a = 2 \cdot (2 \cdot q_1 + r_1) + r_0, \qquad r_1 = 0 \text{ or } 1$$

ie

$$a = 2^2 \cdot q_1 + 2 \cdot r_1 + r_0$$

Now divide q_1 by 2
to get

$$a = 2^2 \cdot (2 \cdot q_2 + r_2) + 2 \cdot r_1 + r_0$$

$$a = 2^3 \cdot q_2 + 2^2 \cdot r_2 + 2 \cdot r_1 + r_0$$

where
$r_2 = 0$ or 1

Keep dividing the quotients by 2,
until you get a quotient of 0.

So we get

$$\boxed{a = r_n \cdot 2^n + r_{n-1} \cdot 2^{n-1} + \cdots + 2^2 \cdot r_2 + 2 \cdot r_1 + r_0}$$

where each $r_i = 0$ or 1, and this
representation is unique, since in the
division algorithm, the remainders were
unique.

$\blacksquare$

Ex:

$1 = 1 \cdot 2^0$

$2 = 1 \cdot 2^1 + 0 \cdot 2^0$

$3 = 2 + 1 = 1 \cdot 2^1 + 1 \cdot 2^0$

$4 = 2^2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$

$5 = 4 + 1 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$

$6 = 4 + 2 = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$

$7 = 6 + 1 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$

$8 = 2^3 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$

$9 = 8 + 1 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$

$\vdots$

Remark: When analyzing a Nim game with a small number of rows and counters, the partial state diagram is a reasonable way of determining the winner. However, for a large number of rows and counters, a more sophisticated method is needed. For this reason we introduce the following definition.

Definition 1: Given a list of whole numbers, the ***nim-sum*** is found by representing the positive numbers as sums of distinct powers of 2, and then canceling all pairs of equal powers, and finally adding what is left. The nim-sum of a list of whole numbers $a_1, a_2, \ldots, a_n$ is denoted by:

$$a_1 \oplus a_2 \oplus \cdots \oplus a_n$$

Note: Whole number here means nonnegative integer.

Example 2: Find the following nim-sums.

a) $2 \oplus 3 = \cancel{2} \oplus (1+\cancel{2}) = 1$

b) $2 \oplus 3 \oplus 4 = \cancel{2} \oplus (1+\cancel{2}) \oplus 4 = 5$

c) $15 \oplus 15 = (\cancel{1}+\cancel{2}+\cancel{4}+\cancel{8}) \oplus (\cancel{1}+\cancel{2}+\cancel{4}+\cancel{8}) = 0$

d) $25 \oplus 31 = (\cancel{1}+\cancel{8}+\cancel{16}) \oplus (\cancel{1}+2+4+\cancel{8}+\cancel{16})$
$= 6$

7

e) $4 \oplus 7 \oplus 12 = \cancel{4} \oplus (1+2+\cancel{4}) \oplus (4+8) = 15$

f) $4 \oplus 7 \oplus \cancel{12} \oplus \cancel{7} \oplus \cancel{12} \oplus \cancel{7} = \cancel{4} \oplus (1+2+\cancel{4}) = 3$

Lemma 1: Given whole numbers a, b, c the following nim-sum properties hold:

1. $a \oplus b$ is a whole number
2. $a \oplus b = b \oplus a$
3. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
4. $a \oplus a = 0$
5. $a \oplus b \neq 0$ whenever $a \neq b$

Proof:

Exercise.

Note: In the $(4,3,1)$ game of nim, • all the winning positions have a nim-sum $\neq 0$.

• all the losing positions have a nim-sum $= 0$.

$4\oplus3\oplus2 = 4\oplus(1+\cancel{2})\oplus\cancel{2} = 5$

$3\oplus2\oplus1 = (\cancel{1}+\cancel{2})\oplus\cancel{2}\oplus\cancel{1} = 0$

$3\oplus1\oplus1 = (\cancel{1}+2)\oplus\cancel{1}\oplus1 = 3$

$3\oplus1\oplus0 = 2$, $2\oplus1\oplus0 = 3$.

$3\oplus2\oplus0 = 1$, $2\oplus1\oplus1 = 2$, $2\oplus2\oplus1 = 1$

$1\oplus1\oplus0 = 0$, $2\oplus2\oplus0 = 0$, $1\oplus0\oplus0 = 1$

$2\oplus0\oplus0 = 2$

Nim is also played in a more general fashion than the version at the beginning of this lecture. Start by picking the number of rows to play with; any amount is fine. In each row place a random amount of counters. Players take turns removing counters provided they are all in the same row. The player that picks up the last counter wins the game.

Definition 2: Similar to the 4, 3, 2 Nim game (at the beginning of this lecture) represent a state of the general Nim game by a list of numbers, one number for each row. Each number will track how many counters remain in its corresponding row. Calculating the nim-sum of these numbers gives the ***nim-sum*** of the game.

Lemma 2: Every move in the game of Nim will change its nim-sum.

Proof. Consider a game of Nim in the state: $a_1, a_2, \dots, a_n$ where $s = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ and suppose decreasing a_1 to a_1' does not change the nim-sum. Then by lemma 1 we have:

$$0 = s \oplus s = (a_1 \oplus a_2 \oplus \cdots \oplus a_n) \oplus (a_1' \oplus a_2 \oplus \cdots \oplus a_n) = a_1 \oplus a_1' \neq 0$$

which is a contradiction. Therefore the nim-sum must change.

Lemma 3: There is always a move that changes a Nim game with a non-zero nim-sum to a zero nim-sum.

Proof. Consider a game of Nim in the state: $a_1, a_2, \dots, a_n$ where $s = a_1 \oplus a_2 \oplus \cdots \oplus a_n \neq 0$. Since s is not zero it can be written as a (non-empty) sum of distinct powers of 2. One of the powers appearing in that sum will be the largest, let it be 2^k. Since 2^k did not cancel when calculating $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ it must have appeared an odd number of times in the sums of distinct powers of 2 of $a_1, a_2, \dots, a_n$. This means we can pick a_i so that 2^k appears in its sum of distinct powers of 2. The move we want to make is to decrease a_i to $s \oplus a_i$ but first we must verify $a_i > s \oplus a_i$ to show this is a valid move. Let's compare the sum of distinct powers of 2 of a_i and the sum of distinct powers of 2 of $s \oplus a_i$. Every power of 2 larger than 2^k appears in both sums of a_i and $s \oplus a_i$. 2^k appears in the sum of a_i but not in the sum of $s \oplus a_i$. Every power of 2 less than 2^k may or may not appear in the sum of a_i and may or may not appear in the sum of $s \oplus a_i$. Now, the following difference is at least:

$$a_i - (s \oplus a_i) = 2^k - 2^{k-1} - 2^{k-2} - \cdots - 2^0 = 2^k - (2^k - 1) = 1$$

therefore, in any case $a_i > s \oplus a_i$. Finally, making the move to decrease a_i to $s \oplus a_i$ changes the nim-sum from:

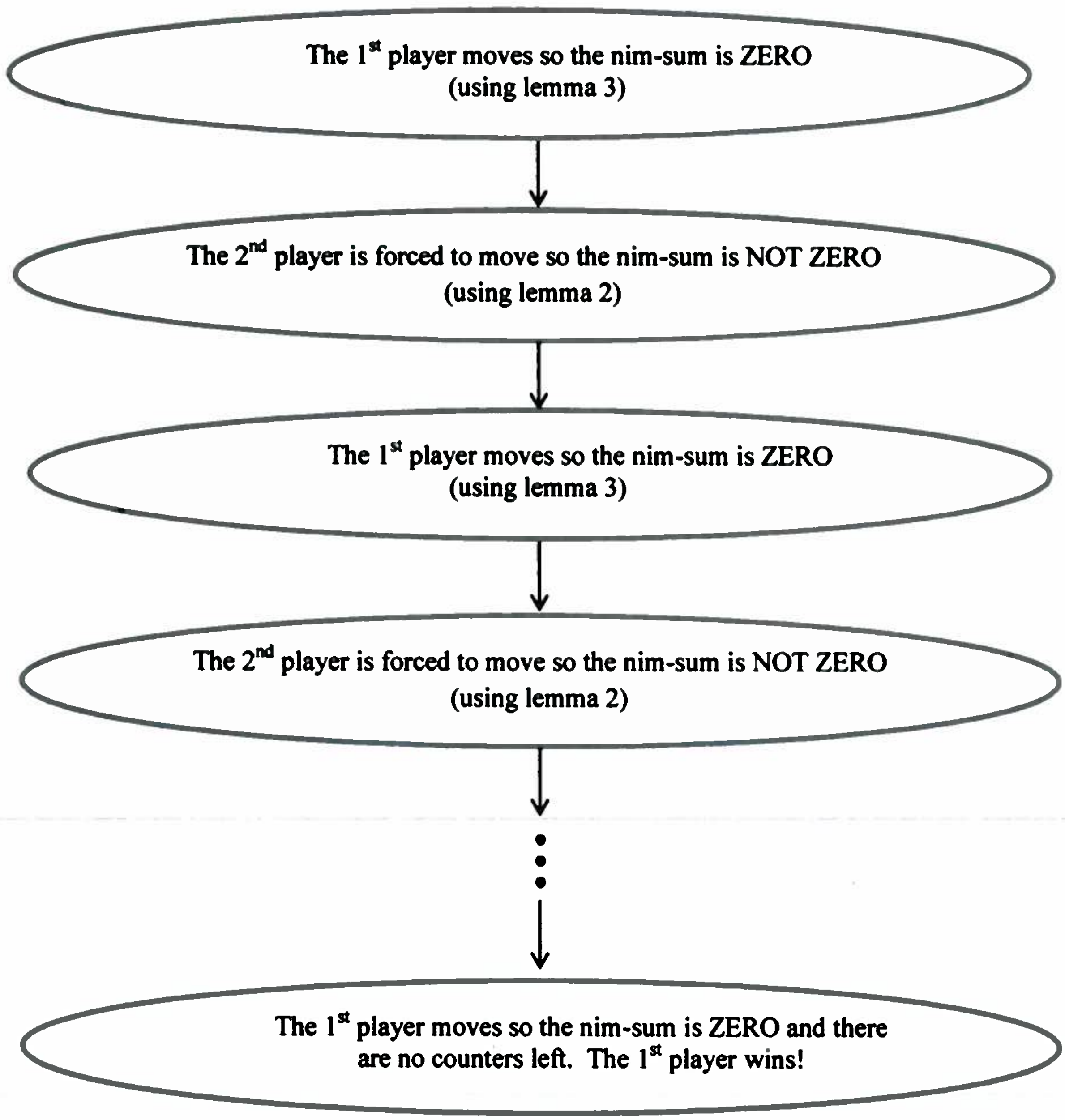
$$a_1 \oplus a_2 \oplus \cdots \oplus \quad a_i \quad \oplus \cdots \oplus a_n \neq 0$$

to

$$\begin{aligned} & a_1 \oplus a_2 \oplus \cdots \oplus (s \oplus a_i) \oplus \cdots \oplus a_n \\ &= (a_1 \oplus a_2 \oplus \cdots \oplus \quad a_i \quad \oplus \cdots \oplus a_n) \oplus s \\ &= s \oplus s \\ &= 0 \end{aligned}$$

Theorem 1: In a Nim game, the first player is in a winning position if and only if the Nim game starts with a non-zero nim-sum.

Proof. Suppose the game starts with a non-zero nim-sum, the first player can use the following winning strategy:



Note: the 2nd player can steal this strategy if the game starts with a nim-sum of zero.

12/

Example 2: In the following Nim games, who can always win, the 1st player or the 2nd player? If the first player can always win, state the first move that he or she needs to make.

a)

$$3 \oplus 4 \oplus 5$$

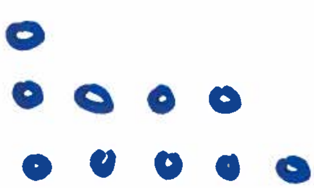
$$= (\cancel{1}+2) \oplus (\cancel{4}) \oplus (1+\cancel{4})$$

$$= 2 \neq 0$$

So the 1st player can always win.

The 1st player picks the row with 3 Counters and decreases it to

$$3 \oplus 2 = (1+2) \oplus 2 = 1$$



$$\cancel{1} \oplus (\cancel{4}) \oplus (\cancel{1}+\cancel{4}) = 0$$

b)

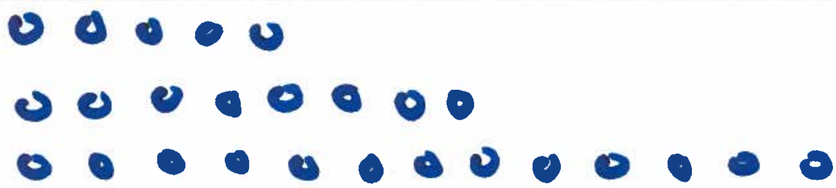
$$5 \oplus 8 \oplus 16 = (1+4) \oplus 8 \oplus 16$$

$$= 29 \neq 0$$

The 1st player can always win and picks the row with 16 and decreases it to

$$16 \oplus 29 = \cancel{16} \oplus (1+4+8+\cancel{16})$$

$$= 13$$

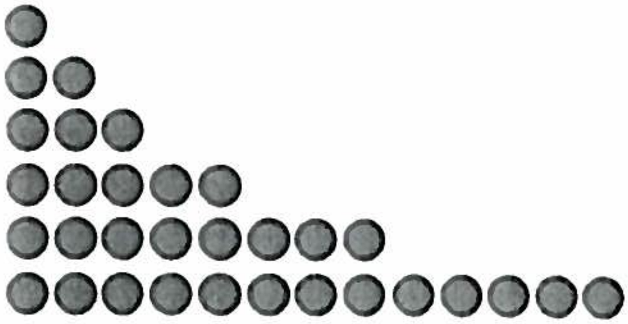


Nim sum

$$5 \oplus 8 \oplus 13 = (\cancel{1}+\cancel{4}) \oplus \cancel{8} \oplus (\cancel{1}+\cancel{4}+\cancel{8})$$

$$= 0$$

c)



$1 \oplus 2 \oplus 3 \oplus 5 \oplus 8 \oplus 13$

$= \cancel{1} \oplus \cancel{2} \oplus (\cancel{1}+\cancel{2}) \oplus (\cancel{1}+\cancel{4}) \oplus \cancel{8} \oplus (\cancel{1}+\cancel{4}+\cancel{8})$

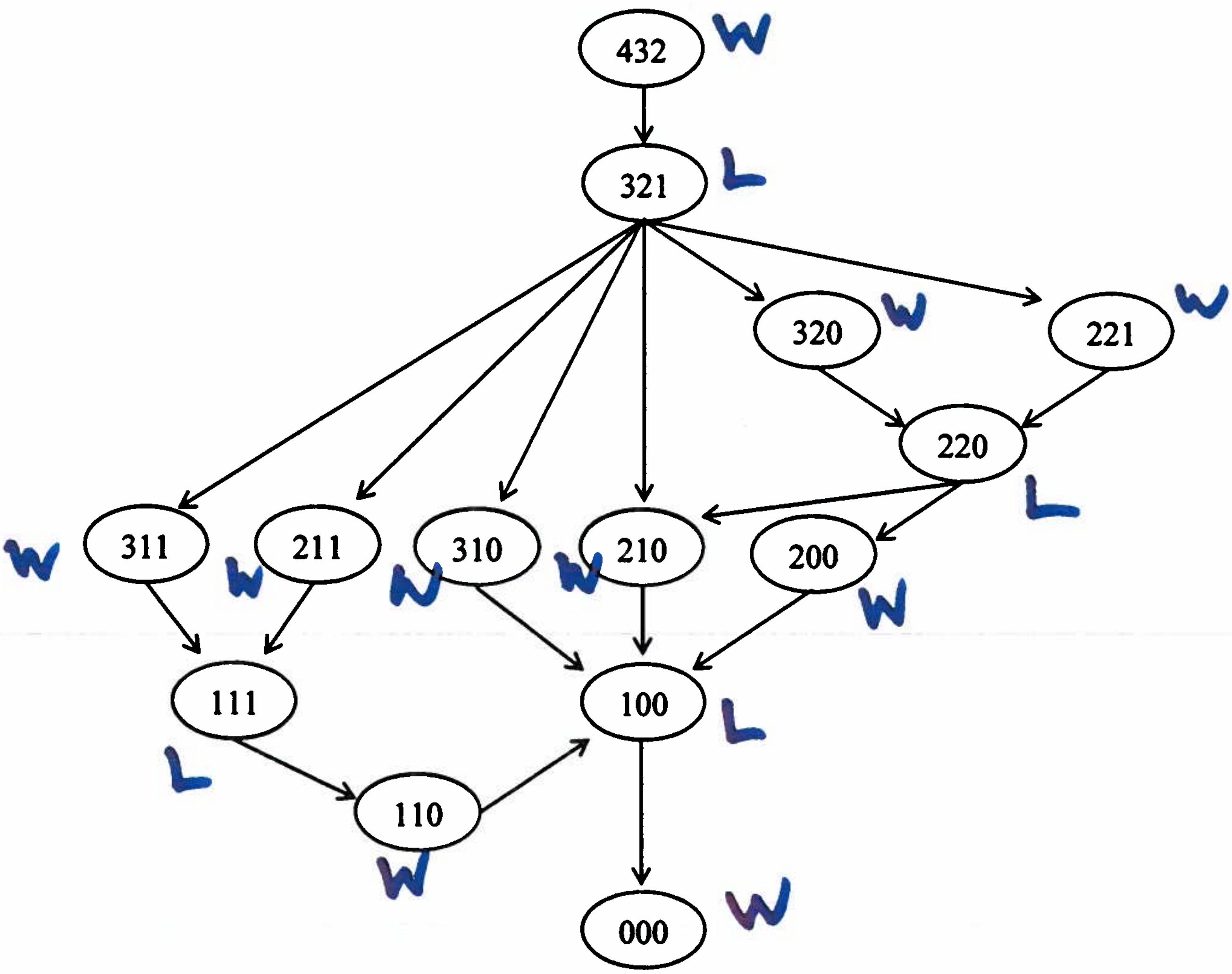
$= 0$

The 2nd player can always win.

Classic Nim

How to play: Place 9 counters as shown. Players take turns removing counters provided they are all in the same row. The player who is forced to pick up the last counter loses.

Use the same representation that was used for Nim and use the following portion of the state diagram to help find the player that can always win.

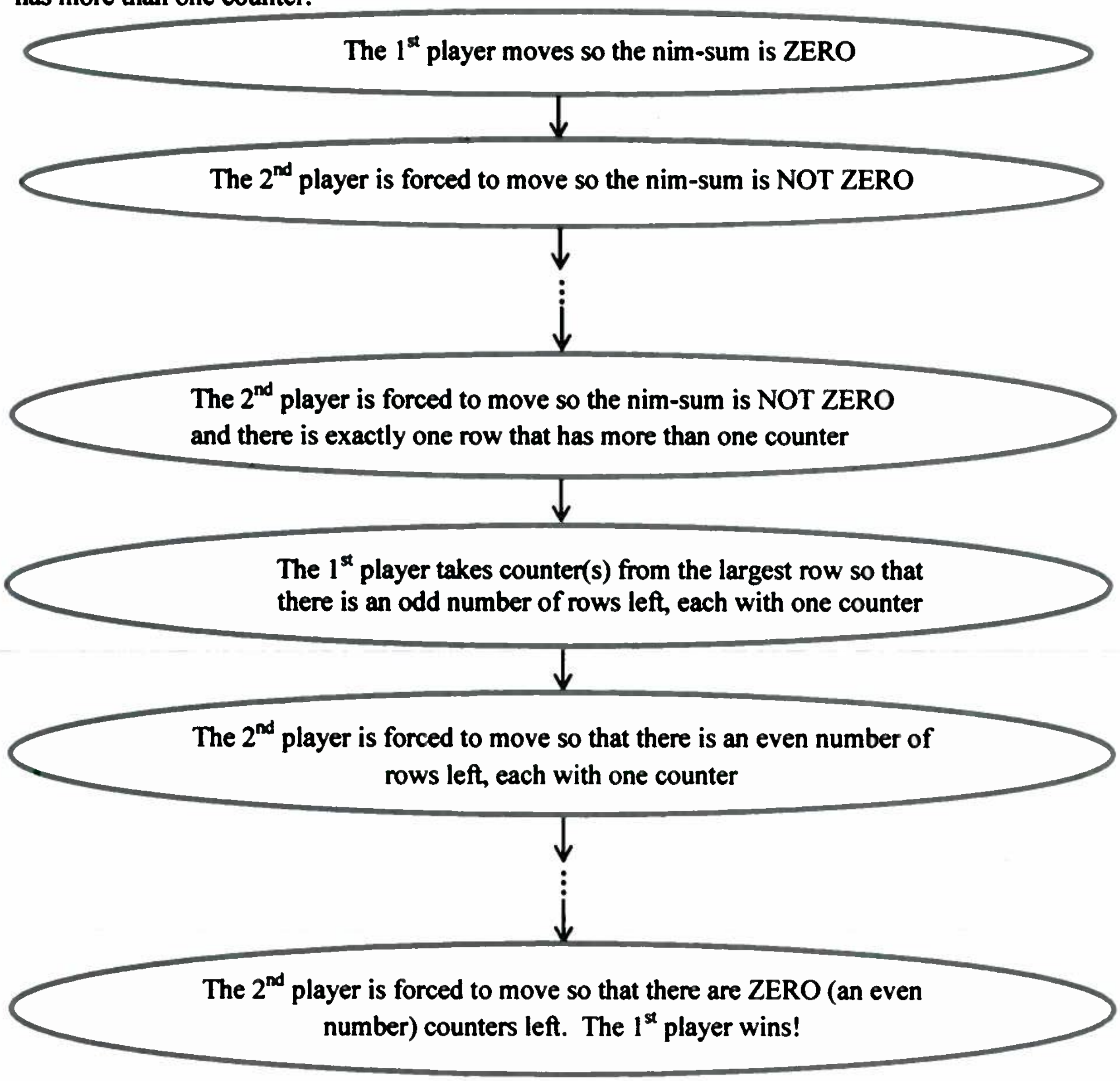


Note: Playing Nim under the rules of Classic Nim is called misère play. Classic Nim can also be played in a more general fashion than the version above. It can be set up in all the same ways as the general version of Nim. The two versions are played in the same way, the only difference is, in Classic Nim, the player forced to pick up the last counter loses.

Remark: Amazingly, the same opening move in both 4, 3, 2 Nim and 4, 3, 2 Classic Nim is successful. To that end, the winning strategy for the general version of Classic Nim is very similar to the winning strategy for the general version of Nim.

Theorem 2: Suppose a Classic Nim game starts with two rows with more than one counter. The first player is in a winning position if and only if the Classic Nim game starts with a non-zero nim-sum.

Proof. If the game starts with a non-zero nim-sum, the first player can use the following winning strategy. This strategy works the same as the one in theorem 1 until there is exactly one row that has more than one counter.



Note: the 2nd player can steal this strategy if the game starts with a nim-sum of zero.

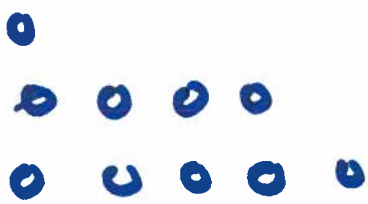
Example 3: In the following Classic Nim games, who can always win, the 1st player or the 2nd player? If the first player can always win, state the first move that he or she needs to make.

a)

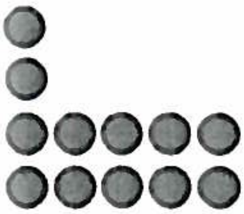
$$3 \oplus 4 \oplus 5 = (\cancel{1}+2) \oplus \cancel{4} \oplus (\cancel{1}+\cancel{4})$$

$$= 2 \neq 0$$

The 1st player can always win.

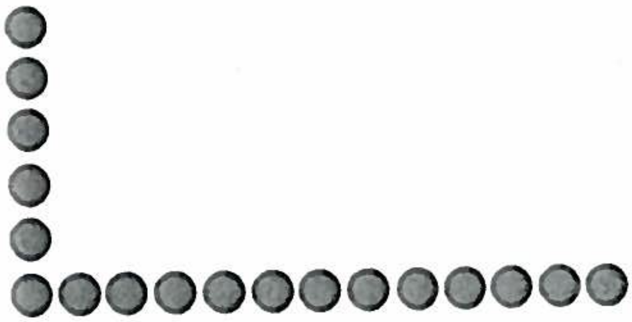


New nim sum is $1 \oplus 4 \oplus 5 = 0$

b) 

$\cancel{1} \oplus \cancel{1} \oplus \cancel{5} \oplus \cancel{5} = 0$

The 2nd playa can always win

c) 

There is exactly one row with more than one counter.

nim-sum

$\cancel{1} \oplus \cancel{1} \oplus \cancel{1} \oplus \cancel{1} \oplus 1 \oplus 13 = \cancel{1} \oplus (\cancel{1} + 4 + 8) = 12 \neq 0$

So the 1st player can always win.

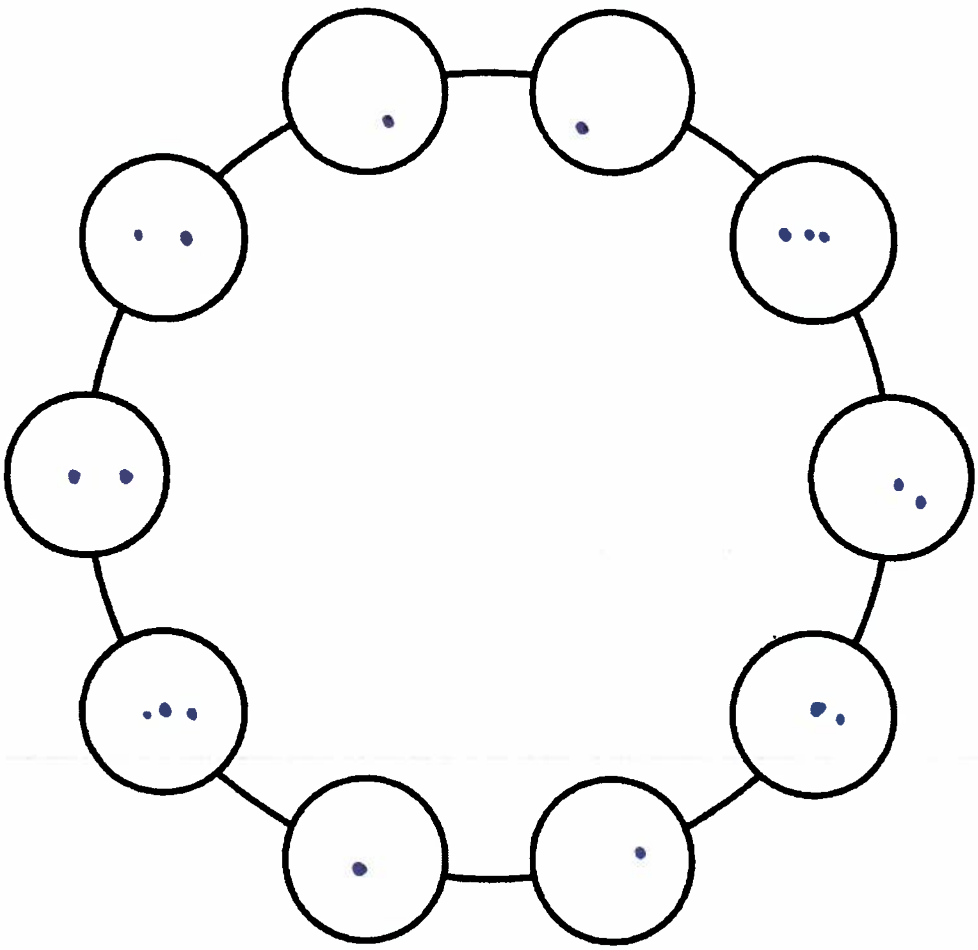
The 1st player takes all of row 6

20/

Circle Nim

This is a two-player game. How to play: To begin, place a marker in each circle. Players take turns removing either one or two markers from the circles. If a player removes two markers, they must be beside each other, with no empty circles between them. The player that removes the last marker(s) wins.

State the player that can always win and describe a winning strategy.



The 2nd player can always win by reflecting the 1st player's move through the center of the circle.