# §Two Player Games

## Lecture 4

**Definition 1**: A strategy that enables a player to win no matter what move his or her opponent makes is called a *winning strategy*.

**Definition 2**: Games of *perfect information* are games in which both players are aware at all times of all aspects of the structure of the game.

**Definition 3**: A *finite game* must necessarily terminate in a finite number of moves.

**Theorem 1**: In any finite two player game of perfect information in which the players move alternately and in which chance does not affect the decision-making process, either

a) one of the two players must have a winning strategy or
b) the game is a theoretical draw.

**Hint**: When looking for a winning strategy, sometimes coloring a checkered pattern on game pieces or game boards can be very helpful.

### Pick up Coins

There are six coins of varying denominations lined up in a row on a table. Nora picks a coin from one of the ends of the row. Next, Abraham picks up a coin from one of the ends of the remaining row of coins. They alternate in this manner until Abraham picks up the last coin. Nora wins the game if she has picked up at least the same amount of money as Abraham.

State the player that can always win and describe a winning strategy.

Checker the 6 coins

Nora can always win by taking the group with the large value.

Ex:

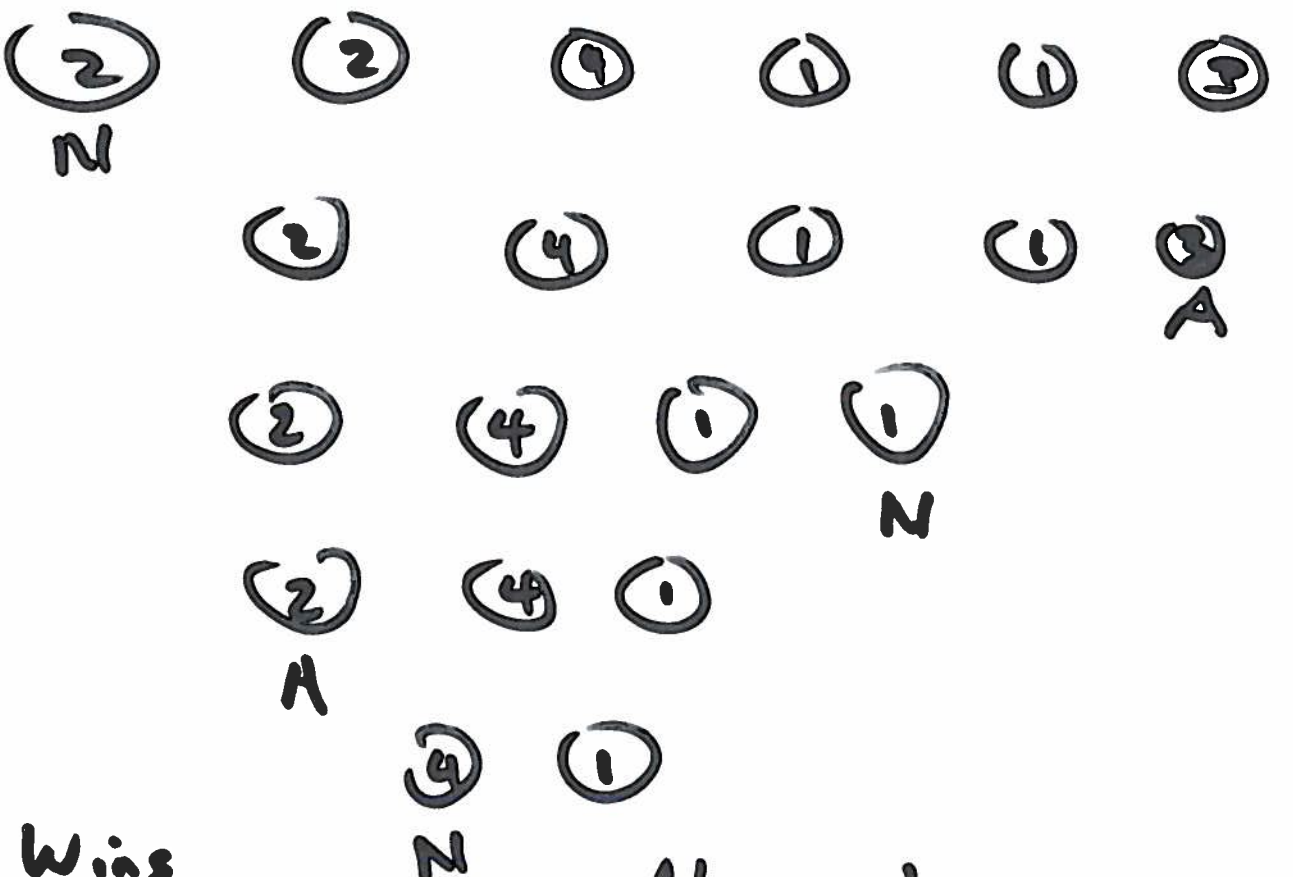

Nora adds the two groups

Group 1: $2+4+1=7$

Group 2: $2+1+3=6$

She takes the coin in group 1.

The game is played as follows.




Nora: Wins

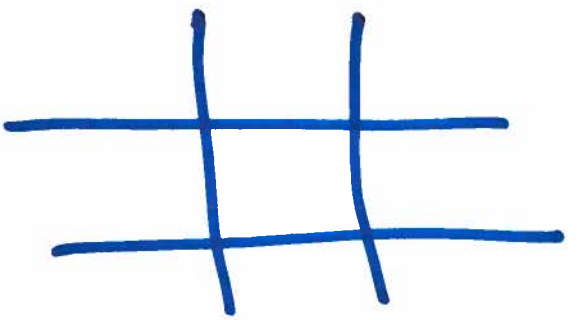
2, 1, 4

Abe: Loses

3, 2, 1

# Tic - Tac - Toe

Player X and Player O alternate turns by placing their symbol in the 3x3 grid

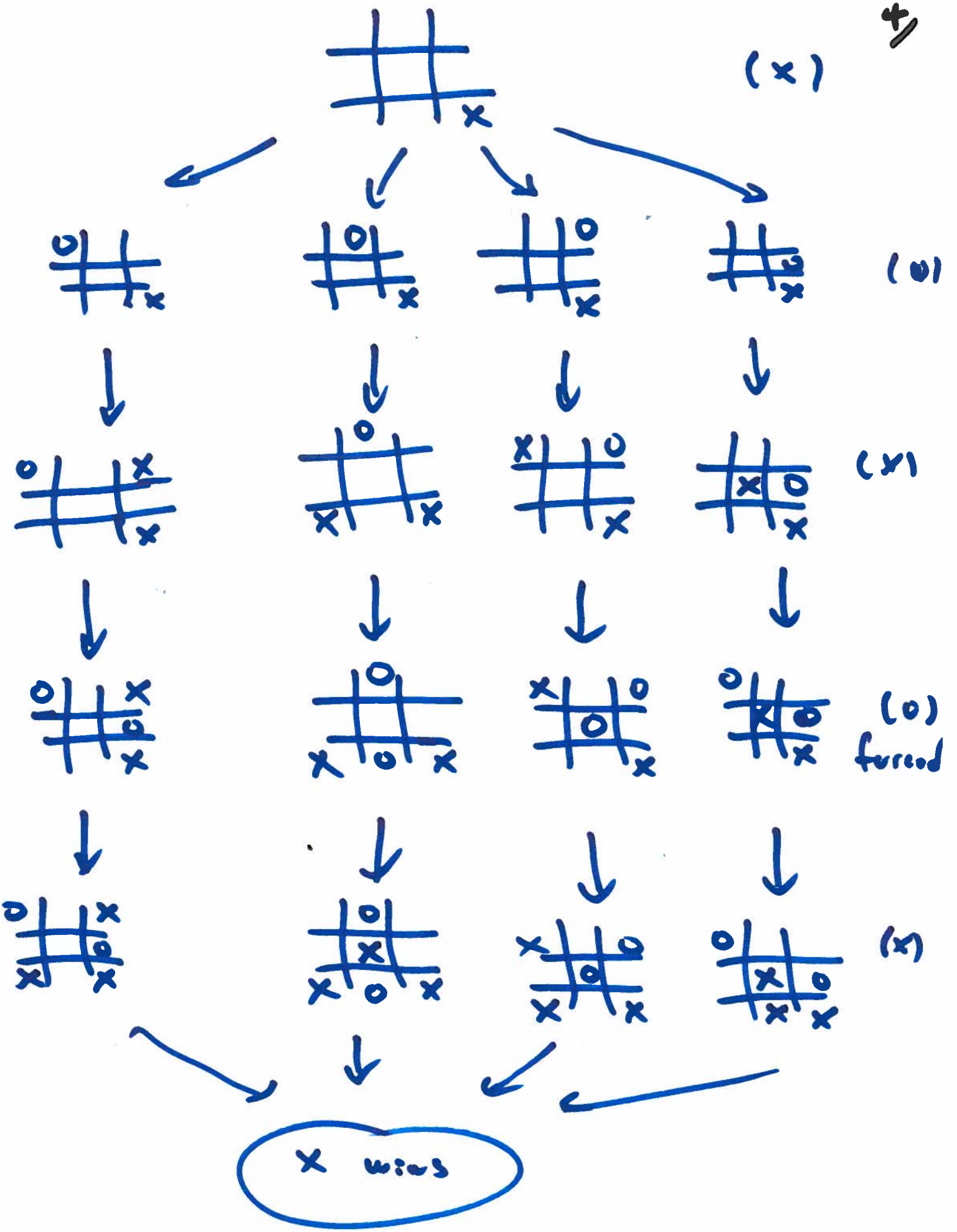


whoever get 3 in a row wins.

Note:

If the first player plays in a corner, then the 2nd player must play in the center.

Draw a tree diagram to show the states of the game.


(x)
(o)
(x)
(o)
forced
(x)
X wins

**Definition 4**: A *state diagram* is a diagram where each vertex represents a position or state of the game. An arrow (or directed edge) is drawn from vertex $P$ to vertex $Q$ if it is possible to move from position $P$ to position $Q$ in one turn of game play. An example of a state diagram is given below for the matchstick game.

**Definition 5**: A player is in a *winning position* if on their turn they can use a winning strategy.

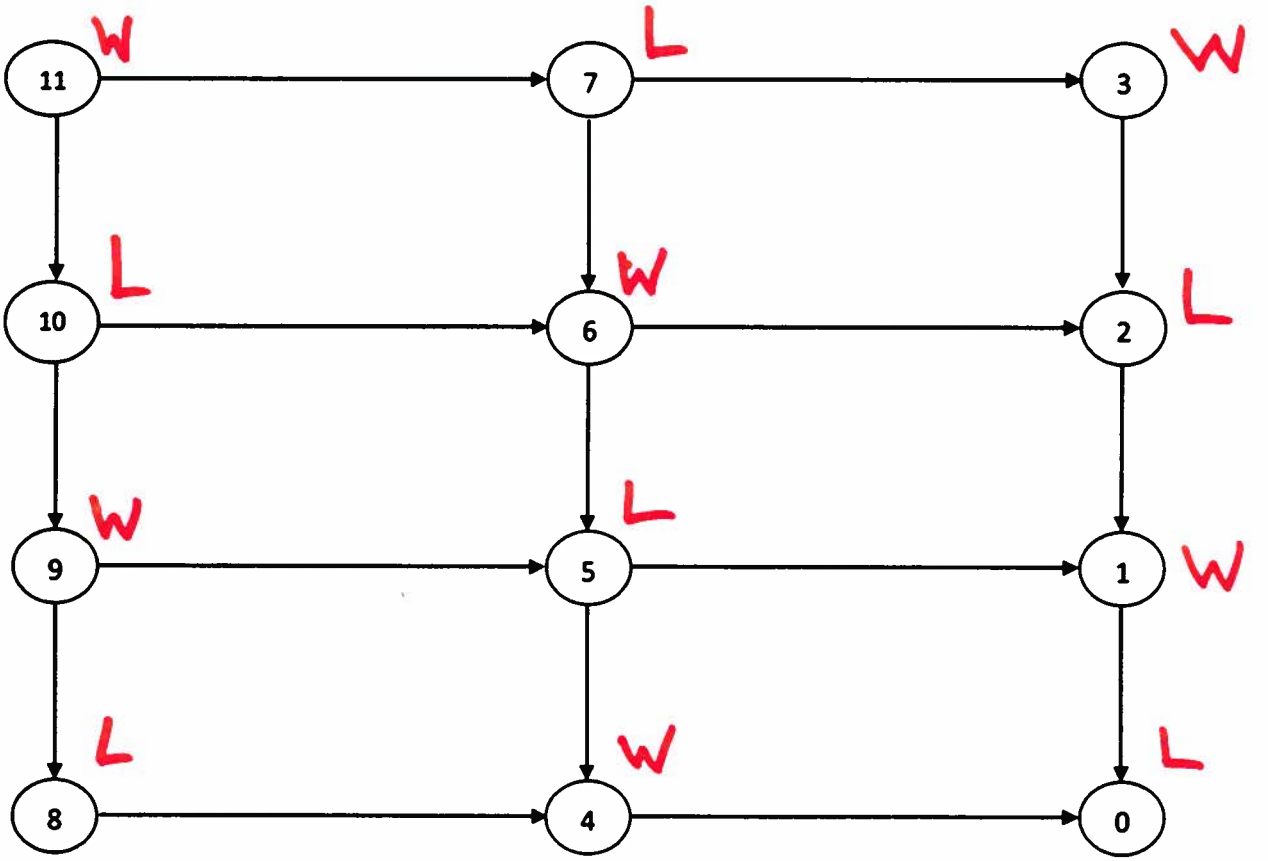
**Definition 6**: A player is in a *losing position* if on their turn they cannot use a winning strategy.

**Hint**: Starting at the end of a state diagram and working backwards is a very helpful way to determine if the starting player is in a winning position or a losing position.

## The Matchstick Game

How to play: from a pile of 11 matchsticks, two players take turns removing either 1 or 4 sticks. There is one exception: if the number of sticks remaining is any multiple of 4, the player must take 4 sticks. The player who removes the last stick is the winner.

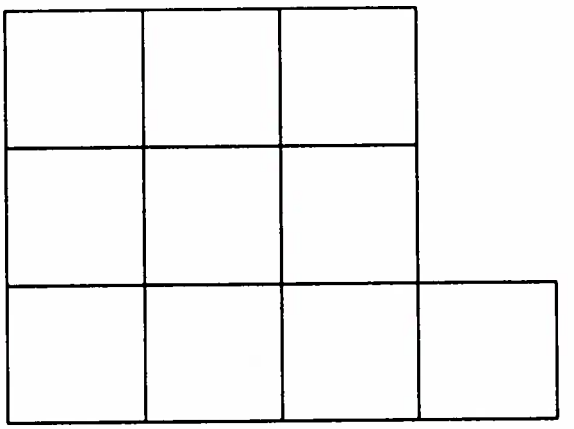
Label each position below as a winning position (W) or a losing position (L). State the player that can always win.




The 1st player always wins!

**Hint**: Often drawing a full state diagram is impractical and unnecessary. A partial state diagram (or a decision tree), if correctly drawn, can also indicate if the first player starts in a wining position or not.

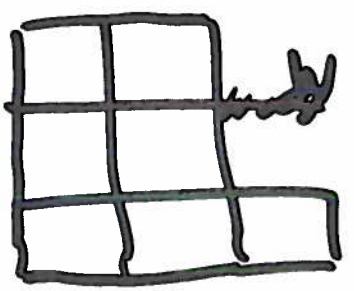
## Ticktack Plus One

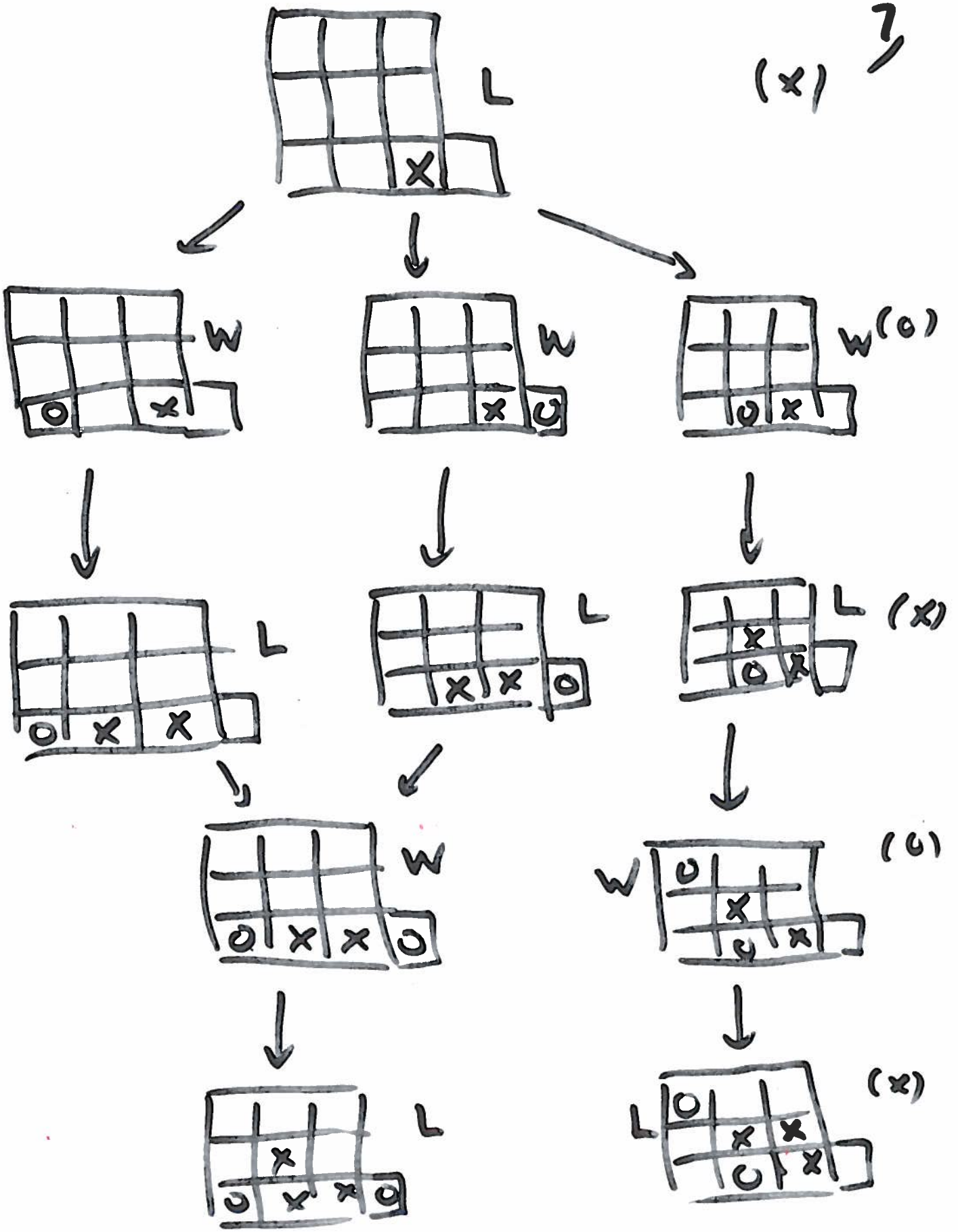


How to play: The game is played in the usual way tick-tack-toe is played but with a single cell added. The first player to get 3 of their markers in a horizontal, vertical, or diagonal row wins the game.

State the player that can always win and describe a winning strategy.

A partial state diagram to show that the 1st player (X) can always win.

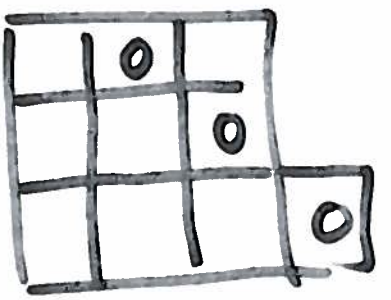





x wins a two more moves.

Note: O can never win

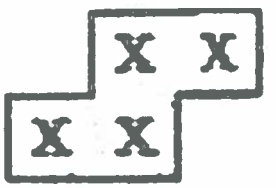
by



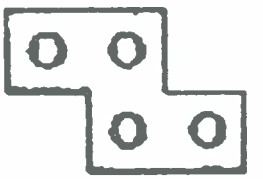
Check this.

**Hint**: When looking for a winning strategy, sometimes mirroring your opponent can be very helpful.

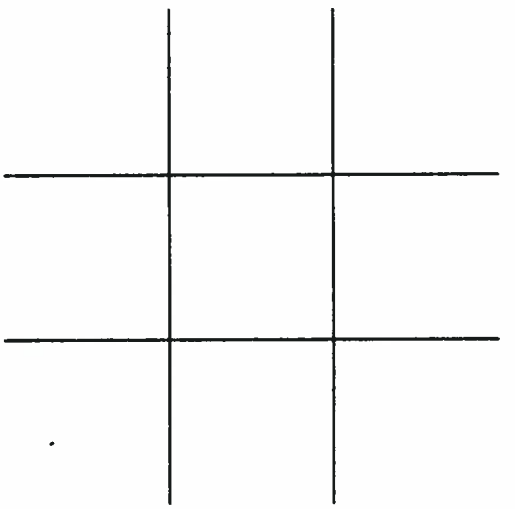
**Hint**: Whenever possible use symmetry when drawing a decision tree.




Snakey's Toe

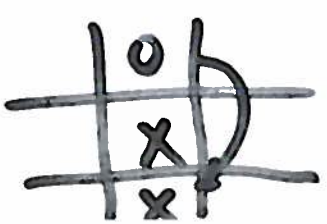



How to play: Player X and Player O alternately put their symbol in an empty square on their turn. Whoever makes Snakey's Toe (shown in the diagrams above) in any direction wins the game.



State the player that can always win and describe a winning strategy.

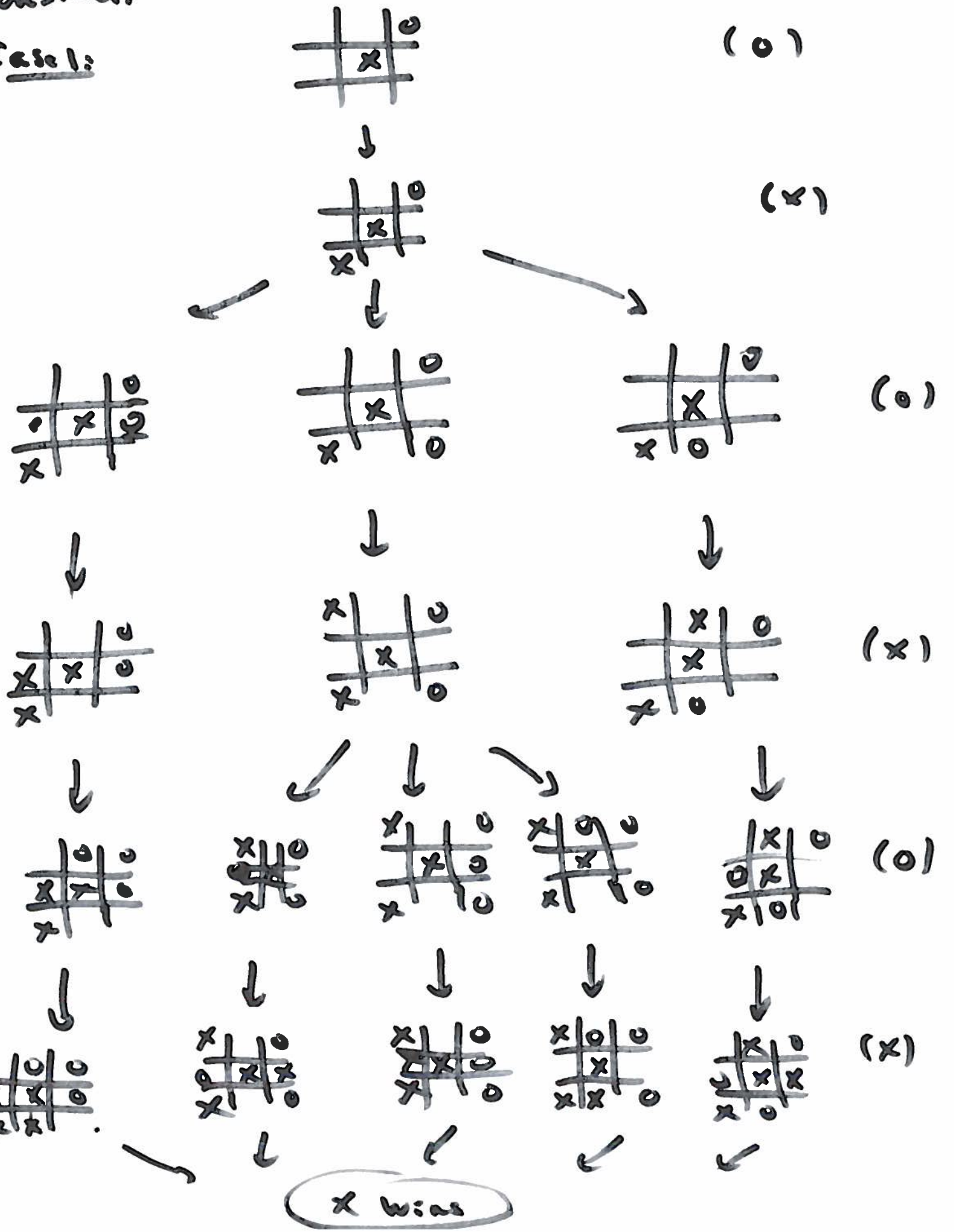
The 1st player X can always win by playing in the center first, and then playing on O's reflection through the center.



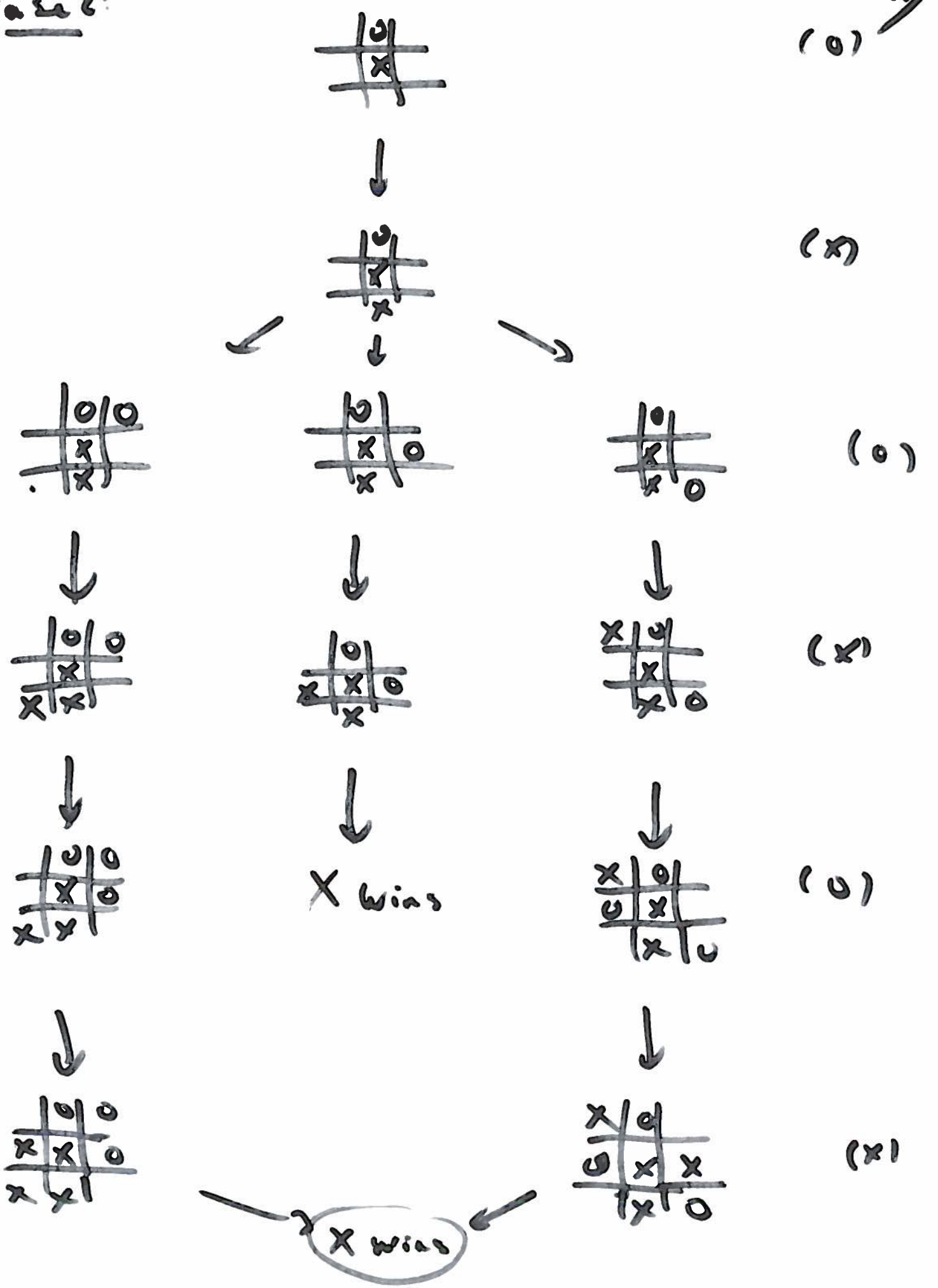


By symmetry only have two cases to Consider. 10/

Case1:

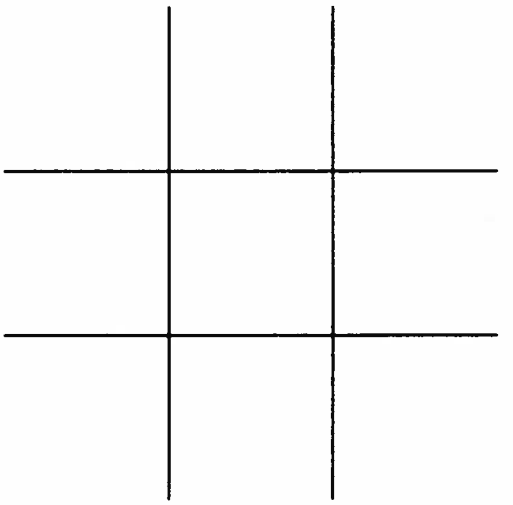


Case 2:



## Wild Tick-Tack-Toe

How to play: In wild tick-tack-toe, you can put either an X or an O in an empty square when it is your turn. Whoever gets three X's or three O's in a row in any direction wins the game.



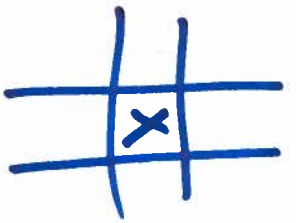
State the player that can always win and describe a winning strategy.

The 1st player can always win, by playing in the center first (either X or O) and then reflecting the 2nd players move through the center.

1st player is A

2nd player is B.

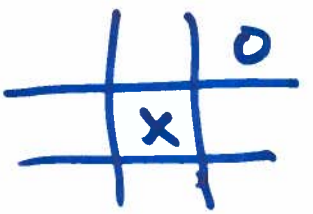
A plays in the center:



(A)

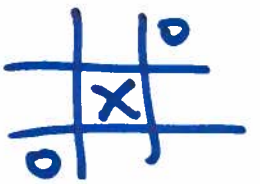
B must play an O.

Case 1: (B plays in a corner)



(B)

↓

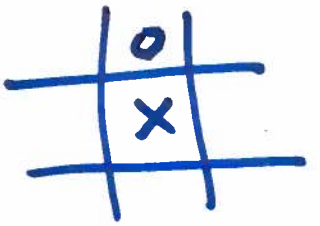


(A)

B cannot play an X or an O

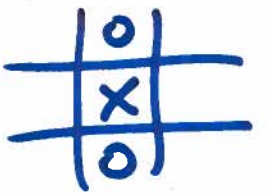
So A wins

Case 3: B plays in the middle of an edge.



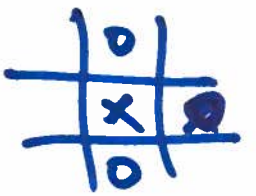
(B)

↓



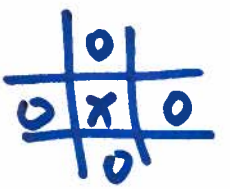
(A)

↓ B is forced



(B)

↓



(A)

B cannot play an x or an o.

So A wins.