Lecture 3

Warm up problem: The Couriers Problem (6.3 Ecco):

There are five parts to a top secret code: A, B, C, D, and E. The code must be sent across enemy territory amongst 8 couriers safely to our agent. The enemy will attack 2 of the couriers receiving the parts of the code they contain. We can let them intercept any four parts of the code but not all 5 parts. By making 3 copies of the code how should the different parts be split amongst the 8 couriers to ensure a safe delivery.

5 parts to code A, B, C, D, E

8 Couriers

Enemy capture up to 2 couriers.

- Need 3 copies of each part to make sure all 5 parts got across.

1 A C ✓		1 A B C ✓		1 ABC ✓
2 A D		2 A D		2 BCD
3 A D		3 A D		3 AD
4 B D	or	4 B D	or	4 AD
5 B E		5 BE		5 BE
6 B E		6 C E		6 C
7 C E		7 C		7 E
8 C		8 E		8 E

Dirichlet's Pigeonhole Principle

If p pigeons enter h pigeonholes and if $p > nh$ for some integer n, then at least one pigeonhole contains more than n pigeons.

Pigeons	Pigeonholes
1	1
2	2
$\vdots$	$\vdots$
p	h

Ex: $h = 3$, $p = 10$

Pigeons

$$10\left\{\begin{matrix} o & o \\ o & o \\ o & o \\ o & o \\ o & o \end{matrix}\right.$$

Holes

$$\left.\begin{matrix} o \\ o \\ o \end{matrix}\right\} 3$$

pick $n = 3$

$p = 10 > 9 = n \cdot h$

One hole has at least 4 pigeons

Theorem: If m pigeons occupy n pigeonholes, then at least one pigeonhole contains

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1$$

pigeons. Here $\left\lfloor \frac{m-1}{n} \right\rfloor$ is the greatest integer $\leq \frac{m-1}{n}$.

Proof: If we had exactly $n \left\lfloor \frac{m-1}{n} \right\rfloor$ pigeons, since

$$n \left\lfloor \frac{m-1}{n} \right\rfloor \leq m-1 < m$$

and we have m pigeons, so at least one pigeonhole has more than $\left\lfloor \frac{m-1}{n} \right\rfloor$ pigeons. $\blacksquare$

Ex: From the previous example

if $m = 10$, and $n = 3$

Then $\left\lfloor \frac{m-1}{n} \right\rfloor = \left\lfloor \frac{9}{3} \right\rfloor = 3$

So that

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = 3 + 1 = 4$$

and at least one pigeonhole contains

4 pigeons.

Warm up problem: The Couriers Problem (6.3 Ecco):

There are five parts to a top secret code: A, B, C, D, and E. The code must be sent across enemy territory amongst 8 couriers safely to our agent. The enemy will attack 2 of the couriers receiving the parts of the code they contain. We can let them intercept any four parts of the code but not all 5 parts. By making 3 copies of the code how should the different parts be split amongst the 8 couriers to ensure a safe delivery.

a) Does a solution exist with 7 couriers?
b) Prove this cannot be done with 6 couriers.
c) Suppose we can split the code up into as many parts as we want; then what is the minimum number of couriers needed?

1 ABC
2 BCD
3 AD
4 AD
5 BE
6 C
7 E
8 E

c) Parts

$$15\left\{\begin{array}{ll} 1 & A \\ 2 & \vdots \\ \vdots & \vdots \\ 15 & E \end{array}\right.$$

Couriers

$$\left.\begin{array}{c} \square \\ \square \\ \square \\ \vdots \\ \square \end{array}\right\} 7$$

$m = 15$, $n = 7$ at least one ~~each~~ p.h. has to contain $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons, i.e.

$$\left\lfloor \frac{14}{7} \right\rfloor + 1 = 3$$

1 ABC
2 DA
3 DA
4 DB
5 EB
6 EC
7 EC

Here D + E must be separate, but can be paired with with one of A, B, C.

(b). Show no soln with 6 Conners.

Again by Pigeonhole principle, if $m = 15$, $n = 6$ then one of Conners carries $\left\lfloor \frac{15-1}{6} \right\rfloor + 1 = \left\lfloor \frac{14}{6} \right\rfloor + 1$

$= 2 + 1 = 3$

But now

1 A B C

2 D

3 D

4 D

5 E

6 E

E?

D + E must be separated, but we can't find a place the 3rd E.

(c). Read soln in Ecco (6.3)

The Pigeonhole Principle:

If p pigeons enter h pigeonholes and if p is greater than nh for some integer n, then at least one pigeonhole contains more than n pigeons.

Ex: Let a, b be positive integers, then decimal expansion of $\frac{a}{b}$ either terminates or repeats.

Terminating: 1.2345

Repeating: 1.2345454545...

$1.23\overline{45}$

Use long division to divide

$a = 138$ by 9

$$\begin{array}{r} 15.3 \\ 9 \overline{)\,138.0000} \\ 9 \\ \hline 48 \\ 45 \\ \hline 30 \\ 27 \\ \hline 30 \ldots \end{array}$$

remainders

4, 3, 3, ...

The sequence of remainders

When we use long division to divide a by b let

$r_0 = a$, and

$r_1, r_2, \ldots, r_k, \ldots$

the successive remainders.

So remainders satisfy

$$0 \le r_k \le b-1$$

If any of the remainders is $0 = r_i$ then the division terminates and $\frac{a}{b}$ has a terminating decimal exp.

If none of the remainders is 0, then the sequence of remainders goes on forever.

So by pigeonhole principle

Some remainder must repeat.

Say $r_j = r_k$ for $j < k$.

So the decimal digits from

divisions between r_j and r_{k-1}

repeat from.

So the decimal expansion of

the rational # $\frac{a}{b}$ repeats.

Example 1: If you pick five different numbers from the integers 1 to 8, show two of them must add up to nine.

Let $a_1, a_2, \dots, a_5 \in \{1, 2, \dots, 8\}$

so $a_i \neq a_j$ if $i \neq j$.

Numbers

$$5\begin{cases} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{cases}$$

Pairs that add to 9

$$\left.\begin{array}{l} \boxed{1 \text{ or } 8} \\ \boxed{2 \text{ or } 7} \\ \boxed{3 \text{ or } 6} \\ \boxed{4 \text{ or } 5} \end{array}\right\} 4$$

One pigeonhole contains 2 pigeons

i.e. there are two distinct numbers

$a_i \neq a_j$ such that $a_i + a_j = 9$

Example 2: Prove that at a party, if certain guests shake hands with certain other guests, there will always be a pair of people who shook the same number of hands. (It is possible that someone didn't shake hands with anyone.)

Suppose there are $n \geq 2$ guests.

Case 1: Some one shook no hands.

Then no one shook everybody's hand.

So each guest shook

$$\underbrace{0, 1, 2, \ldots, n-2}_{n-1} \text{ hands}$$

Case 2: No one shook no hands

So each guest shook either

$$\underbrace{1, 2, 3, \ldots, n-1}_{n-1} \text{ hands}$$

Guests	Choices for # of hands shaken
$n \left\{ \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \vdots \\ \bullet \end{array} \right.$	$\left. \begin{array}{c} \square \\ \square \\ \vdots \\ \square \end{array} \right\} n-1$

By P.H.P. at least 2 guests shake same # of hands

Example 3: Prove that in a collection of $n + 1$ distinct integers, there are distinct integers x and y such that $x - y$ is a multiple of n.

Let $x_1, x_2, \dots, x_{n+1}$ be distinct integers.

$$n+1\left\{\begin{array}{c}\underline{\text{Integers}}\\ x_1\\ x_2\\ \vdots\\ x_{n+1}\end{array}\right. \qquad\qquad \left.\begin{array}{c}\underline{\text{mod } n}\\ 0\\ 1\\ 2\\ \vdots\\ n-1\end{array}\right\}n$$

Therefore from P.H.P. there are integers x_i and x_j such that

$$x_i \equiv x_j \pmod{n}$$

so $$x_i - x_j \equiv 0 \pmod{n}$$

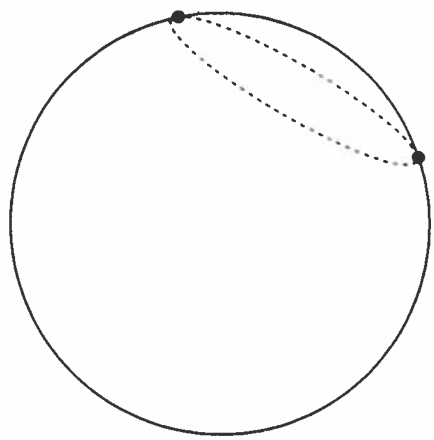
So n divides $x_i - x_j$, i.e.

$$x_i - x_j = k \cdot n \quad \text{for some integer } k.$$

Example 4: If you pick five points on the surface of a sphere, then there is a way to slice the sphere with a plane so that four of the points will lie on the same side of the plane (suppose a point exactly on the slice belongs to both sides of the slice).

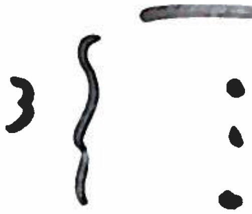
Pick any two points on the sphere, and slice the sphere with a plane passing through only these two of the five points. The intersection of the plane and the sphere is a circle on the surface of the sphere.

Pick any 2 points



Assume that these two points belong to both parts.

3 points left



by P.H.P. one piece has 2 more pts on it. So one of the pieces has 4 pts on it.

Math Party:

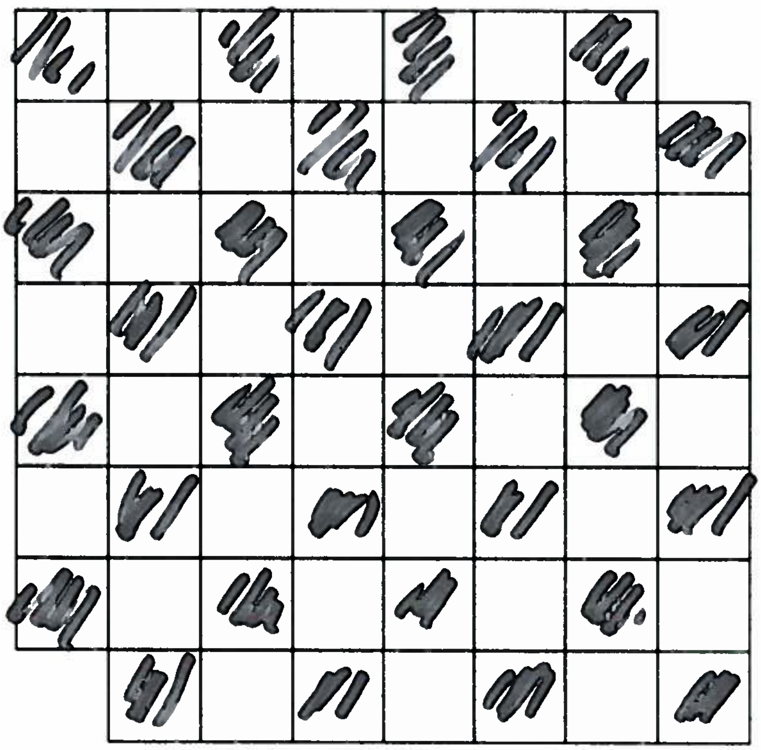
Wed 1:00 – 3:00

SAB 325

- Answer Questions about
  - Course
  - H.W.
  - Exams.
  - ??

Example 5:

a) Given an 8 by 8 grid of squares, with two opposite corners missing, can you cover the grid with dominos? Why or why not? A domino takes up exactly two board squares.



Checker the grid

Have 32 Black squares

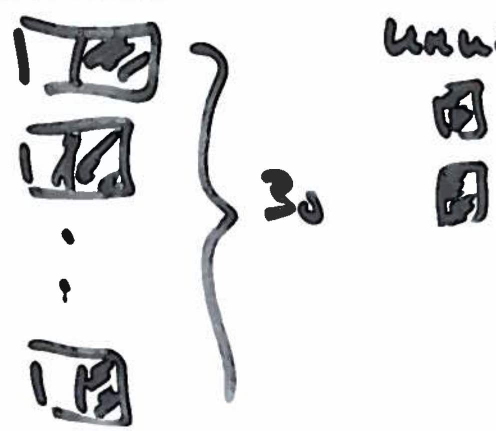
and 30 white squares

Domino's

31

1x2 Squares

30



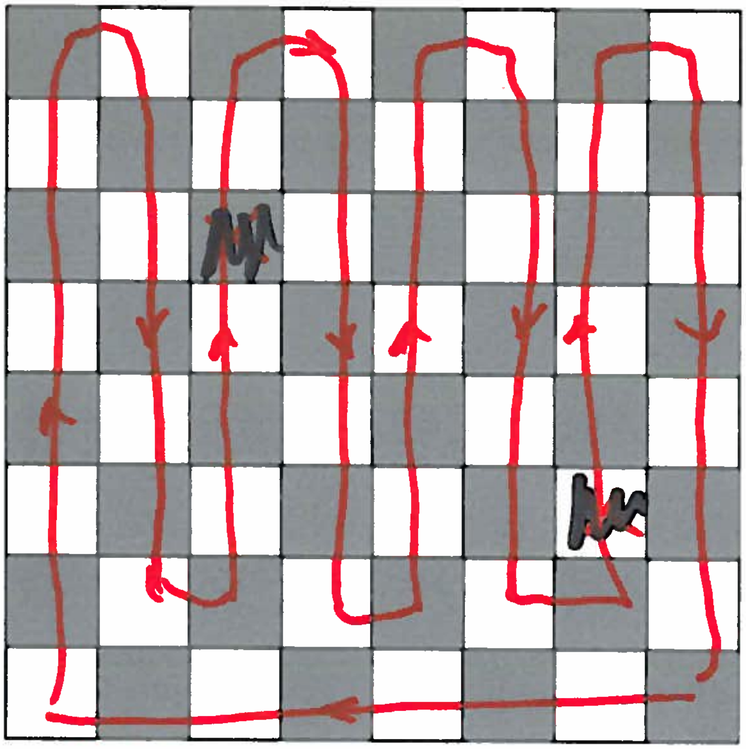
unused

31 dominoes are needed, can cover at most 30 pigeonholes

Answer is No! Can't do it

40/

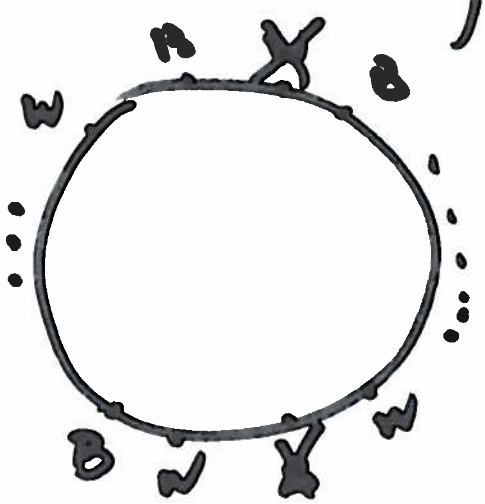
b) Given a checker board (8 by 8) if any two squares of opposite color are missing can the board be covered with dominos?



Yes!

1. Make a maze which cycles through all the squares.

2. Remove any 1 W Sqr. and any 1 B Sqr.



3. Have pieces which can be covered with dominos.

Example 6: Trevor is trying to fatten up to fit into his tuxedo before an important day. Over a 30 day period, he pledges to eat bacon at least once per day, and 45 times in all. Show there will be a period of consecutive days where he eats bacon exactly 14 times.

Let S_i = # of times Kevin ate bacon on the 1st i days.

S_0

$$0 < S_1 < S_2 < S_3 < \cdots < S_{30} = 45$$

$$14 < S_1 + 14 < S_2 + 14 < \cdots < S_{30} + 14 = 59$$

Sums/Cumm Sums	Range
$60\left\{\begin{array}{l}\text{distinct}\left\{\begin{array}{l} S_1 \\ S_2 \\ \vdots \\ S_{30}\end{array}\right. \\ \text{distinct}\left\{\begin{array}{l} S_1+14 \\ S_2+14 \\ \vdots \\ S_{30}+14\end{array}\right.\end{array}\right.$	$\left.\begin{array}{c} 1 \\ 2 \\ \vdots \\ 30 \\ \vdots \\ 59\end{array}\right\} 59$

One value in the range is equal to at least 2 of $S_1, S_2, \ldots, S_{30}, S_1+14, S_2+14, \ldots S_{30}+14$

Since $s_1, s_2, \dots s_{30}$ are all different

then $s_1+14, s_2+14, \dots, s_{30}+14$ all different.

So $s_i = s_j + 14$

for some $i > j$

Therefore from day j to day i

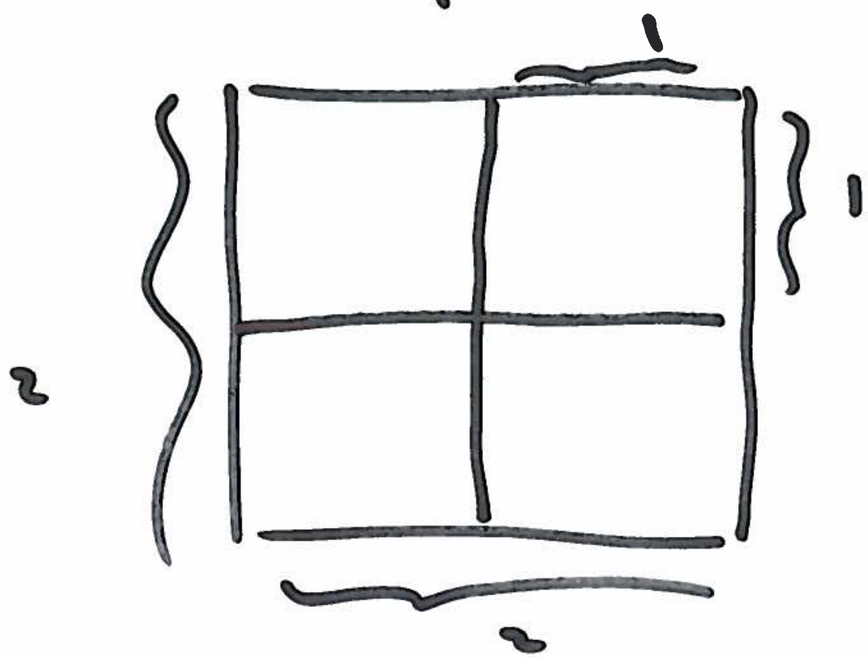
Trevor eats bacon exactly 14

times

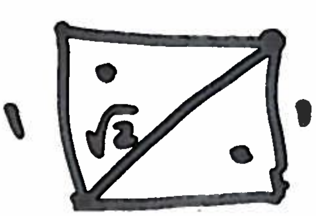
Example 7: Show that by placing 5 points anywhere within a square of side length 2, there will surely be a pair of points that are distance $\sqrt{2}$ or less from each other.

Disect square into 4 parts

4 1x1 squares



One square contains at least two points by P.H.P.



So dist $\leq \sqrt{2}$