Intro. Lecture 1

Example 8-coin problem

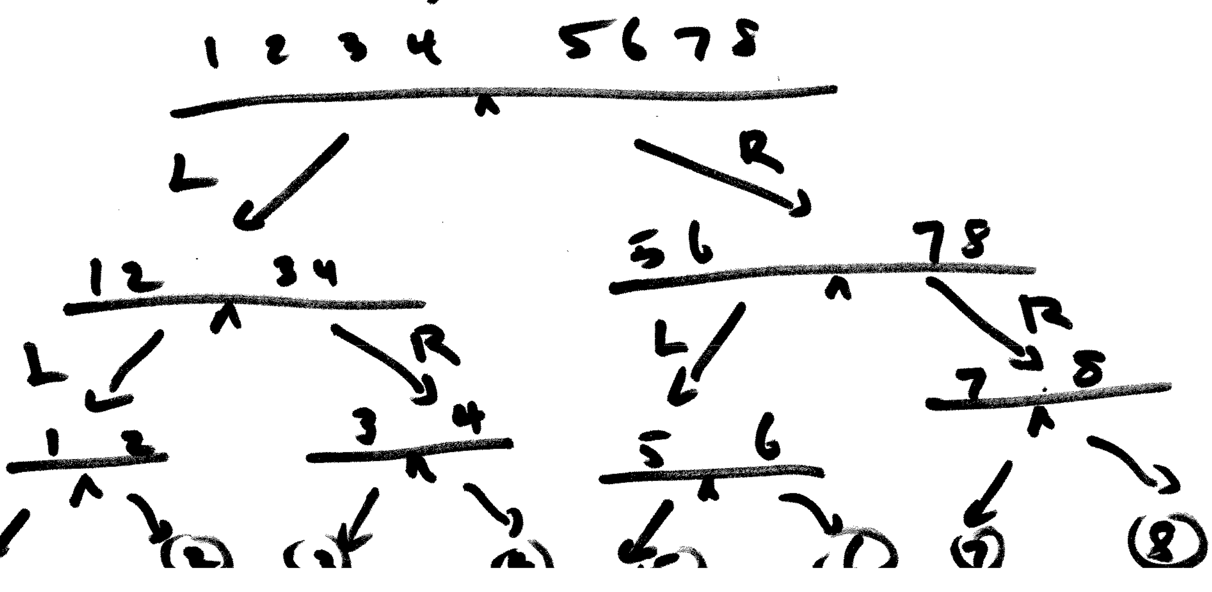
Given 8 seemingly identical coins. One is fake and weighs slightly more than the others. The real coins are all exactly the same weight.

Find the fake coin in as few weighings as possible.

- In 3 weighings:

Use a pan balance: 0000 0000

Similar to binary search:



Can we do it with fewer weighings?

Yes! Split into 3 piles of 3, 3, and 2 coins.

1st Weighing: Weigh 3 against 3 leaving the pile of 2 coins aside. This tells which of the three piles has the heavy coin.

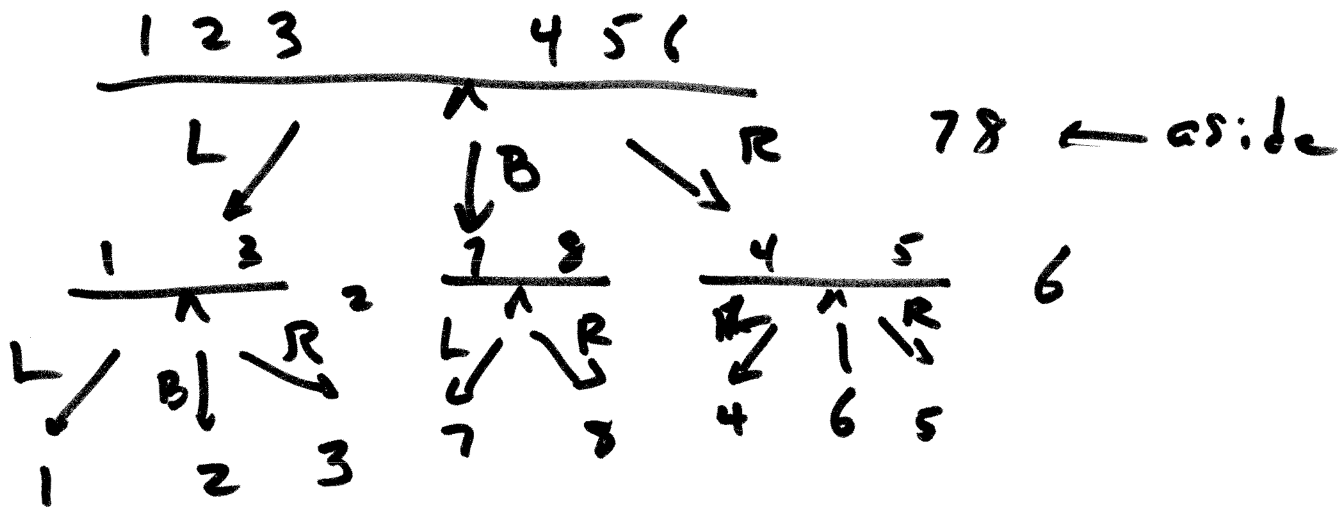
2nd Weighing: Compare the coins in the heavy pile. If the fake is in a pile of 3, weigh one against one, leaving one aside. This reveals which is the heavy coin.

This is an example of an adaptive solution.

- An *adaptive* *solution*: each step depends on previous steps.

Example 1: Adaptive soln to 8 coin problem.

L - left
R - right
B - balance



Can't do it with 1 weighing.

- A **nonadaptive solution** is a fixed set of steps leading to the solution.

136 | 247   5,8

146 | 238   5,7

125 | 348   67

Heavy		Light	Fake
LLL	or	RRR	1
RRL	or	LLR	2
LRR	or	RLL	3
RLR	or	LRL	4
BBL	or	BBR	5
LLB	or	RRB	6
RBB	or	LBB	7
BRR	or	BLL	8

↑

possible outcomes

Result 1: Using n weighings on a pan balance we can find 1 counterfeit coin (know it is Heavier) amongst 3^n coins.

proof:

Each weighing has 3 outcomes: L, B, R

- n Weighings can spell 3^n distinct words of length n, using letters L, B, R ∎

Ex: $n = 3$

$$\left.\begin{array}{ll} LLL & RRR \\ LLR & RRL \\ LRL & RLR \\ \vdots & \\ BLL & BRR \\ BBB & \end{array}\right\} 3^3 = 27 \text{ words.}$$

Note:

- Each word can point to a counterfeit coin, but a words' complement (flip R's and L's) can point to the same coin.

Result 2: Using n weighings it may be possible to find one counterfeit coin (heavier H or L) among at most $\left\lceil \frac{3^n}{2} \right\rceil$

Notation:

If $x \in \mathbb{R}$, $\text{floor}(x) = \lfloor x \rfloor$ = greatest integer $\leq x$.

$\text{Ceiling}(x) = \lceil x \rceil$ = smallest integer $\geq x$.

<u>Card Trick</u>: 8 cards

A, B, C, D, E, F, G, H

arranged in 3 columns

A	B	C
D	E	F
G	H	

ask which row card is in.

→ A	D	G
B	E	H
C	F	

ask which row card is in

- Can you solve the 9-coin problem knowing it is Heavy in 2 weighings? Yes!

A B C
D E F } weigh
G H I } aside

A D G
B E H } weigh
C F I } aside

Q. How about 10 coins?

- With 10 coins, the 1st weighing divides them into 3 piles (two for the pans, and a pile left over - the latter may be empty).

At least one of these piles has 4 coins (since $3+3+3=9<10$)

Suppose the fake coin is among those 4. The best the next weighing can do is separate the 4 into 3 piles. This leaves 2 in one pile, and if the fake is one of the two we need a third weighing.

Keystone Kidnapper:

1	2	3	4	5	6	7
13	12	11	10	9	8	
	14	15	16	17	18	19
25	24	23	22	21	20	
	26	27	28	29	30	31
37	36	35	34	33	32	

Key m 54321.

Key is m. $X \equiv 54321 \pmod{12}$