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Properties of the Integers

The set of all integers is the set

$$\mathbb{Z} = \{\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots\},$$

and the subset of $\mathbb{Z}$ given by

$$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\},$$

is the set of *nonnegative integers* (also called the *natural numbers* or the *counting numbers*).

We assume that the notions of addition $(+)$ and multiplication $(\,\cdot\,)$ of integers have been defined, and note that $\mathbb{Z}$ with these two binary operations satisfy the following.

Axioms for Integers

- **Closure Laws:** if $a, b \in \mathbb{Z}$, then

  $$a + b \in \mathbb{Z} \qquad \text{and} \qquad a \cdot b \in \mathbb{Z}.$$

- **Commutative Laws:** if $a, b \in \mathbb{Z}$, then

  $$a + b = b + a \qquad \text{and} \qquad a \cdot b = b \cdot a.$$

- **Associative Laws:** if $a, b, c \in \mathbb{Z}$, then

  $$(a + b) + c = a + (b + c) \qquad \text{and} \qquad (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

- **Distributive Law:** if $a, b, c \in \mathbb{Z}$, then

  $$a \cdot (b + c) = a \cdot b + a \cdot c \qquad \text{and} \qquad (a + b) \cdot c = a \cdot c + b \cdot c.$$

- **Identity Elements:** There exist integers 0 and 1 in $\mathbb{Z}$, with $1 \neq 0$, such that

  $$a + 0 = 0 + a = a \qquad \text{and} \qquad a \cdot 1 = 1 \cdot a = a$$

  for all $a \in \mathbb{Z}$.

- **Additive Inverse:** For each $a \in \mathbb{Z}$, there is an $x \in \mathbb{Z}$ such that

  $$a + x = x + a = 0,$$

  x is called the **additive inverse** of a or the **negative** of a, and is denoted by $-a$.

The set $\mathbb{Z}$ together with the operations of $+$ and $\cdot$ satisfying these axioms is called a **commutative ring with identity**.

We can now prove the following results concerning the integers.

Theorem. For any $a \in \mathbb{Z}$, we have $0 \cdot a = a \cdot 0 = 0$.

Proof. We start with the fact that $0 + 0 = 0$. Multiplying by a, we have

$$a \cdot (0 + 0) = a \cdot 0$$

and from the distributive law we have,

$$a \cdot 0 + a \cdot 0 = a \cdot 0.$$

If $b = -(a \cdot 0)$, then

$$(a \cdot 0 + a \cdot 0) + b = a \cdot 0 + b = 0,$$

and from the associative law,

$$a \cdot 0 + (a \cdot 0 + b) = 0,$$

that is,

$$a \cdot 0 + 0 = 0,$$

and finally,

$$a \cdot 0 = 0.$$

□

Theorem. For any $a \in \mathbb{Z}$, we have $-a = (-1) \cdot a$.

Proof. Let $a \in \mathbb{Z}$, then

$$0 = 0 \cdot a = [1 + (-1)] \cdot a = 1 \cdot a + (-1) \cdot a,$$

so that

$$-a + 0 = -a + (a + (-1) \cdot a),$$

that is,

$$-a = (-a + a) + (-1) \cdot a,$$

that is,

$$-a = 0 + (-1) \cdot a,$$

and finally, $-a = (-1) \cdot a$. □

Theorem. $(-1) \cdot (-1) = 1$.

Proof. We have

$$(-1) \cdot (-1) + (-1) = (-1) \cdot (-1) + (-1) \cdot 1 = (-1) \cdot \big[(-1) + 1\big] = (-1) \cdot 0 = 0,$$

so that

$$\big[(-1) \cdot (-1) + (-1)\big] + 1 = 0 + 1 = 1,$$

that is,

$$(-1) \cdot (-1) + \big[(-1) + 1\big] = 1,$$

or,

$$(-1) \cdot (-1) + 0 = 1.$$

Therefore, $(-1) \cdot (-1) = 1$. □

We can define an ordering on the set of integers $\mathbb{Z}$ using the set of positive integers $\mathbb{N}^+ = \{1,\, 2,\, 3,\, \cdots\}$.

Definition. If $a, b \in \mathbb{Z}$, then we define $a < b$ if and only if $b - a \in \mathbb{N}^+$.

Note: By $b-a$ we mean $b+(-a)$, and if $a < b$ we also write $b > a$. Also, we note that a is a positive integer if and only if $a > 0$, since by definition $a > 0$ if and only if $a = a - 0 \in \mathbb{N}^+$.

Order Axioms for the Integers

- **Closure Axioms for** $\mathbb{N}^+$: If $a, b \in \mathbb{N}^+$, then

$$a + b \in \mathbb{N}^+ \qquad \text{and} \qquad a \cdot b \in \mathbb{N}^+.$$

- **Law of Trichotomy:** For every integer $a \in \mathbb{Z}$, exactly one of the following is true:

$$a \in \mathbb{N}^+ \qquad \text{or} \qquad -a \in \mathbb{N}^+ \qquad \text{or} \qquad a = 0.$$

Exercise. Use the Law of Trichotomy together with the fact that $(-1)\cdot(-1) = 1$ to show that $1 > 0$.

Definition. We say that an integer a is a **zero divisor** or **divisor of zero** if and only if $a \neq 0$ and there exists an integer $b \neq 0$ such that $a \cdot b = 0$.

Now we can show that $\mathbb{Z}$ with the usual notion of addition and multiplication has no zero divisors.

Theorem. If $a, b \in \mathbb{Z}$ and $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Proof. Suppose that $a, b \in \mathbb{Z}$ and $a \cdot b = 0$. If $a \neq 0$ and $b \neq 0$, since

$$a \cdot b = (-a)\cdot(-b) \qquad \text{and} \qquad -a \cdot b = (-a)\cdot b = a \cdot (-b),$$

by considering all possible cases, the fact that $\mathbb{N}^+$ is closed under multiplication and the Law of Trichotomy imply that $a \cdot b \neq 0$, which is a contradiction. Therefore, if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. ☐

Thus, $\mathbb{Z}$ with the usual notion of addition and multiplication is a commutative ring with identity which has no zero divisors, such a structure is called an **integral domain**, and we have the following result.

Theorem. (Cancellation Law) If $a, b, c \in \mathbb{Z}$ with $c \neq 0$, and if $a \cdot c = b \cdot c$, then $a = b$.

Proof. If $a \cdot c = b \cdot c$, then $(a - b)\cdot c = 0$, and since $c \neq 0$, then $a - b = 0$. ☐

Exercise. Show that the relation on $\mathbb{Z}$ defined by $a \leq b$ if and only if $a < b$ or $a = b$, is a **partial ordering**, that is, it is

- **Reflexive:** For each $a \in \mathbb{Z}$, we have $a \leq a$.
- **Antisymmetric:** For each $a, b \in \mathbb{Z}$, if $a \leq b$ and $b \leq a$, then $a = b$.
- **Transitive:** For each $a, b, c \in \mathbb{Z}$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

Show also that this is a **total ordering**, that is, for any $a, b \in \mathbb{Z}$, either $a \leq b$ or $b \leq a$.

We have the standard results concerning the order relation on $\mathbb{Z}$. We will prove (ii), (iv), and (v), and leave the rest as exercises.

Theorem. If $a, b, c, d \in \mathbb{Z}$, then

(i) if $a < b$, then $a \pm c \leq b \pm c$.

(ii) If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$.

(iii) If $a < b$ and $c < 0$, then $a \cdot c > b \cdot c$.

(iv) If $0 < a < b$ and $0 < c < d$, then $a \cdot c < b \cdot d$.

(v) If $a \in \mathbb{Z}$ and $a \neq 0$, then $a^2 > 0$. In particular, $1 > 0$.

Proof.

(ii) If $a < b$ and $c > 0$, then $b - a > 0$ and $c > 0$, so that $(b - a) \cdot c > 0$, that is, $b \cdot c - a \cdot c > 0$. Therefore, $a \cdot c < b \cdot c$.

(iv) We have

$$b \cdot d - a \cdot c = b \cdot d - b \cdot c + b \cdot c - a \cdot c = b \cdot (d - c) + c \cdot (b - a) > 0$$

since $b > 0,\ c > 0,\ d - c > 0,$ and $b - a > 0$.

(v) Let $a \in \mathbb{Z}$, if $a > 0$, then (ii) implies that $a \cdot a > a \cdot 0$, that is, $a^2 > 0$.

If $a < 0$, then $-a > 0$, and (ii) implies that $a^2 = (-a) \cdot (-a) > 0$. Finally, since $1 \neq 0$, then $1 = 1^2 > 0$.
$\square$

Exercise. Show that if $a, b, c \in \mathbb{Z}$ and $a \cdot b < a \cdot c$ and $a > 0$, then $b < c$.

Finally, we need one more axiom for the set of integers.

Well-Ordering Axiom for the Integers

If B is a nonempty subset of $\mathbb{Z}$ which is bounded below, that is, there exists an $n \in \mathbb{Z}$ such that $n \leq b$ for all $b \in B$, then B has a smallest element, that is, there exists a $b_0 \in B$ such that $b_0 < b$ for all $b \in B$, $b \neq b_0$.

In particular, we have

Theorem. (Well-Ordering Principle for $\mathbb{N}$)

Every nonempty set of nonnegative integers has a least element.

It can be shown that the Well-Ordering Principle for $\mathbb{N}$ is logically equivalent to the Principle of Mathematical Induction, so we may assume one of them as an axiom and prove the other one as a theorem.

Exercise. Show that the following statement is equivalent to the Well-Ordering Axiom for the Integers:

> Every nonempty subset of integers which is bounded above has a largest element.

Example. The set of **rational numbers**

$$\mathbb{Q} = \big\{a/b \;\big|\; a, b \in \mathbb{Z},\, b \neq 0\big\}$$

with the usual ordering is not a well–ordered set, that is, there exists a nonempty subset B of $\mathbb{Q}$ which is bounded below, but which has no smallest element.

Proof. In fact, we can take $B = \mathbb{Q}^+$, the set of all positive rational numbers; clearly $\mathbb{Q}^+ \neq \emptyset$ and $0 < q$ for all $q \in \mathbb{Q}^+$, so it is also bounded below. Now, suppose that $\mathbb{Q}^+$ has a smallest element, say $q_0 \in \mathbb{Q}^+$, then $q_0/2 \in \mathbb{Q}^+$ also, and $q_0/2 < q_0$, which is a contradiction. Therefore, our original assumption must have been false, and $\mathbb{Q}^+$ has no smallest element, so $\mathbb{Q}$ is not well–ordered. $\square$

Definition. The set of **irrational numbers** is the set of all real numbers that are not rational, that is, the set $\mathbb{R} \setminus \mathbb{Q}$.

Example. The real number $\sqrt{2}$ is irrational.

Proof. We will show this using the Well-Ordering Principle. First note that the integer 2 lies between the squares of two consecutive positive integers (consecutive squares), namely, $1 < 2 < 4$, and therefore

$$1 < \sqrt{2} < 2,$$

(since $0 < \sqrt{2} \leq 1$ implies $2 \leq 1$, a contradiction; while $\sqrt{2} \geq 2$ implies $2 \geq 4$, again, a contradiction).

Now let

$$B = \{b \in \mathbb{N}^+ \mid \sqrt{2} = a/b \text{ for some } a \in \mathbb{Z}\},$$

if $\sqrt{2} \in \mathbb{Q}$, then $B \neq \emptyset$.

Since B is bounded below by 0, then the Well-Ordering Principle implies that B has a smallest element, call it b_0, so that

$$\sqrt{2} = \frac{a_0}{b_0}$$

where $a_0, b_0 \in \mathbb{N}^+$, and $2b_0^2 = a_0^2$. Since

$$1 < \frac{a_0}{b_0} < 2,$$

then $b_0 < a_0 < 2b_0$, and therefore $0 < a_0 - b_0 < b_0$.

Now we find a positive integer x such that

$$\frac{x}{a_0 - b_0} = \frac{a_0}{b_0},$$

that is, $b_0 x = a_0(a_0 - b_0) = a_0^2 - a_0 b_0 = 2b_0^2 - a_0 b_0 = b_0(2b_0 - a_0)$, so we may take $x = 2b_0 - a_0$, and

$$\sqrt{2} = \frac{2b_0 - a_0}{a_0 - b_0} = \frac{a_0}{b_0},$$

so that $a_0 - b_0 \in B$, and $0 < a_0 - b_0 < b_0$. However, this contradicts the fact that b_0 is the smallest element in B, so our original assumption is incorrect. Therefore, $B = \emptyset$ and $\sqrt{2}$ is irrational. $\square$

Exercise. Show that if m is a positive integer which is not a perfect square, that is, m is not the square of another integer, then $\sqrt{m}$ is irrational.

Hint: The proof mimics the proof above for $\sqrt{2}$.

Definition. If $n \in \mathbb{Z}$, then we say that n is **even** if and only if there exists an integer $k \in \mathbb{Z}$ such that $n = 2k$. We say that n is **odd** if and only if there is an integer $k \in \mathbb{Z}$ such that $n = 2k + 1$.

We will use the Well-Ordering Principle to show that every integer is either even or odd, but first we need a lemma.

Lemma. There does not exist an integer n satisfying $0 < n < 1$.

Proof. Let
$$B = \{n \mid n \in \mathbb{Z}, \text{ and } 0 < n < 1\}.$$
If $B \neq \emptyset$, since B is bounded below by 0, then by the Well-Ordering Principle B has a smallest element, say $n_0 \in B$, but then multiplying the inequality $0 < n_0 < 1$ by the positive integer n_0, we have
$$0 < n_0^2 < n_0 < 1.$$
However, n_0^2 is an integer and so $n_0^2 \in B$, which contradicts the fact that n_0 is the smallest element of B. Therefore, our original assumption is incorrect and $B = \emptyset$, that is, there does not exist an integer n satisfying $0 < n < 1$. Note that we have shown that 1 is the smallest positive integer. $\square$

Theorem. Every integer $n \in \mathbb{Z}$ is either even or odd.

Proof. Suppose there exists an integer $N \in \mathbb{Z}$ such that N is neither even nor odd, let
$$B = \{n \in \mathbb{Z} \mid n \text{ is even or odd and } n \leq N\},$$
then $B \neq \emptyset$ and B is bounded above by N. By the Well-Ordering Property, B has a largest element, say $n_0 \in B$. Since n_0 is either even or odd, and $n_0 \leq N$, then we must have the strict inequality $n_0 < N$.

If n_0 is even, then $n_0 + 1$ is odd, and since n_0 is the largest such integer in B, then we must have
$$n_0 < N < n_0 + 1.$$
If n_0 is odd, then $n_0 + 1$ is even, and again, since n_0 is the largest such integer in B, we must have
$$n_0 < N < n_0 + 1.$$
Thus, in both cases, $N - n_0$ is an integer and
$$0 < N - n_0 < 1,$$
which is a contradiction. Therefore, our original assumption was incorrect, and there does not exist an integer $N \in \mathbb{Z}$ which is neither even nor odd, that is, every integer $n \in \mathbb{Z}$ is either even or odd. $\square$

Theorem. There does not exist an integer $a \in \mathbb{Z}$ which is both even and odd. Thus the set of integers $\mathbb{Z}$ is partitioned into two disjoint classes, the even integers and the odd integers.

Proof. Suppose that $a \in \mathbb{Z}$ and a is both even and odd, then there exist $k, \ell \in \mathbb{Z}$ such that
$$a = 2k \qquad \text{and} \qquad a = 2\ell + 1,$$
and therefore $2\ell + 1 = 2k$, so that $2(k - \ell) = 1$.

Now, since $1 > 0$, the law of trichotomy implies that $k - \ell > 0$. Also, since $2 = 1 + 1 > 1 + 0 = 1$, then

$$1 = 2 \cdot (k - \ell) > 1 \cdot (k - \ell) = k - \ell.$$

Therefore, $k - \ell$ is an integer satisfying $0 < k - \ell < 1$, which is a contradiction, and our assumption that there exists an integer a which is both even and odd is false. $\square$

Greatest Integer Function

Definition. For each real number x, the **greatest integer less than or equal to** x, denoted by $[x]$, is the unique integer n such that

$$n \leq x \leq n + 1.$$

The greatest integer function is also called the **floor function** and is denoted by $\lfloor x \rfloor$.

For $x \in \mathbb{R}$, the existence of $[x]$ follows directly from the well-ordering axiom, since if we let

$$B = \{n \in \mathbb{Z} \,|\, n \leq x\},$$

then $B \neq \emptyset$ and B is bounded above by x, thus B has a smallest element.

As an application, we prove the following result due to Dirichlet.

Theorem. (Pigeonhole Principle)

If $m, n \in \mathbb{Z}^+$, with $m > n$, and if m pigeons occupy n pigeonholes, then at least one pigeonhole has

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1$$

or more pigeons roosting in it.

Proof. The largest multiple of n less than m is found by dividing $m - 1$ by n and discarding the fractional part, that is,

$$\left\lfloor \frac{m-1}{n} \right\rfloor.$$

To see this, we want the largest positive integer k such that $kn < m$, that is, $kn \leq m-1$, but $(k+1)n > m-1$. Thus, we want

$$k \leq \frac{m-1}{n} < k + 1,$$

and therefore $k = \left\lfloor \frac{m-1}{n} \right\rfloor$.

If we had exactly

$$n \left\lfloor \frac{m-1}{n} \right\rfloor \leq m - 1 < m$$

pigeons, we could put $\left\lfloor \frac{m-1}{n} \right\rfloor$ in each pigeonhole. But, since we have m pigeons, at least one pigeonhole contains more than this number of pigeons. $\square$

Note that if $n \in \mathbb{Z}$ and $m = n + 1$, then

$$\left\lfloor \frac{m-1}{n} \right\rfloor = 1,$$

and we have the original statement due to Dirichlet:

Pigeonhole Principle If $n + 1$ or more objects are placed into n boxes, then at least one box contains two or more of the objects.