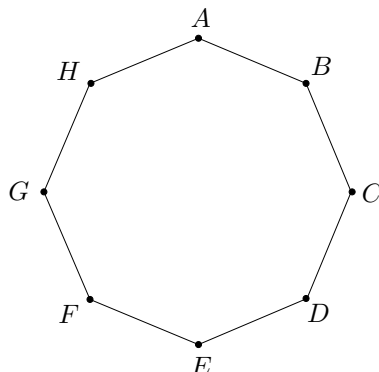


## Frog Problem.

Let  $A$  and  $E$  be opposite vertices of a regular octagon. A frog starts jumping at vertex  $A$ . From any vertex of the octagon except  $E$ , it may jump to either of the two adjacent vertices. When it reaches vertex  $E$ , the frog stops and stays there. Let  $a_n$  be the number of distinct paths of exactly  $n$  jumps ending at  $E$ .



Show that  $a_{2n-1} = 0$  for  $n = 1, 2, \dots$ , while

$$a_{2n} = \frac{1}{\sqrt{2}} \left[ (2 + \sqrt{2})^{n-1} - (2 - \sqrt{2})^{n-1} \right],$$

for  $n = 1, 2, \dots$ .

*Note:* A path of  $n$  jumps is a sequence of vertices  $\{P_0, \dots, P_n\}$  such that

- (i)  $P_0 = A$ , and  $P_n = E$ ,
- (ii) for every  $i$ ,  $0 \leq i \leq n-1$ ,  $P_i$  is distinct from  $E$ ,
- (iii) for every  $i$ ,  $0 \leq i \leq n-1$ ,  $P_i$  and  $P_{i+1}$  are adjacent.

**SOLUTION:** The frog cannot reach vertex  $E$  of the octagon  $ABCDEFGH$  in less than 4 jumps, so that  $a_1 = a_2 = a_3 = 0$ . However, he can reach  $E$  in 4 jumps by going in either direction, so that  $a_4 = 2$ . An odd number of jumps from  $A$  will get the frog to  $B$ ,  $D$ ,  $F$ , or  $H$ , but never to  $E$ ; therefore  $a_{2n-1} = 0$  for all  $n \geq 1$ .

Now let  $b_n$  be the number of distinct paths of exactly  $n$  jumps starting at  $C$  and ending at  $E$  (this is the same as the number starting at  $G$ ). There is just one way to get from  $C$  to  $E$  in 2 jumps, so that  $b_2 = 1$ . After the first 2 jumps from  $A$ , the frog is either at  $C$ , or at  $G$ , or back at  $A$ . He can get back to  $A$  in 2 ways:

$$A \rightarrow B \rightarrow A \quad \text{or} \quad A \rightarrow H \rightarrow A$$

so the number  $a_n$  of paths from  $A$  to  $E$  is made up of the number of paths of  $n-2$  jumps from  $C$ , from  $G$ , and twice the number from  $A$ . Thus

$$a_n = 2b_{n-2} + 2a_{n-2}. \quad (*)$$

for all  $n > 2$ .

When the frog starts at  $C$  and makes more than two jumps, the first two either leave him at  $A$  or back at  $C$  (since landing at  $E$  would mean he made exactly two jumps). Therefore, for  $n > 2$ , the number  $b_n$  of paths from  $C$  to  $E$  is made up of the number of paths of  $n - 2$  jumps from  $A$  and twice the number of paths of  $n - 2$  jumps from  $C$ , since the first 2 jumps could have been

$$C \rightarrow D \rightarrow C \quad \text{or} \quad C \rightarrow B \rightarrow C.$$

Therefore,

$$b_n = 2b_{n-2} + a_{n-2} \quad (**)$$

for all  $n > 2$ .

Subtracting (\*) from (\*\*), we have  $b_n - a_n = -a_{n-2}$ , and replacing  $n$  by  $n - 2$ , we get  $b_{n-2} = a_{n-2} - a_{n-4}$  for  $n > 4$ . Substituting this into (\*), we get

$$a_n = 4a_{n-2} - 2a_{n-4} \quad (***)$$

for  $n > 4$ , and  $a_1 = a_2 = a_3 = 0$ ,  $a_4 = 2$ .

We already know that  $a_{2n-1} = 0$  for all  $n \geq 1$ , and if we put  $A_n = a_{2n}$ , for  $n > 2$ , then  $A_n$  satisfies the difference equation

$$A_n = 4A_{n-1} - 2A_{n-2} \quad (+)$$

for  $n > 2$ , together with the initial conditions

$$A_1 = 0 \quad A_2 = 2. \quad (++)$$

Assuming a solution to (+) of the form  $A_n = \lambda^n$ , we get the characteristic equation

$$\lambda^2 - 4\lambda + 2 = 0$$

with distinct real roots given by

$$\lambda_1 = 2 + \sqrt{2} \quad \text{and} \quad \lambda_2 = 2 - \sqrt{2},$$

so that the general solution to (\*) can be written in the form  $A_n = c_1\lambda_1^n + c_2\lambda_2^n$ .

Using the initial conditions (++) the constants  $c_1$  and  $c_2$  must satisfy the linear equations

$$\begin{aligned} \lambda_1 c_1 + \lambda_2 c_2 &= 0 \\ \lambda_1^2 c_1 + \lambda_2^2 c_2 &= 2, \end{aligned}$$

whose unique solution is given by

$$c_1 = \frac{\sqrt{2} - 1}{2} = \frac{1}{\sqrt{2}}(2 + \sqrt{2})^{-1}$$

and

$$c_2 = -\frac{\sqrt{2} + 1}{2} = -\frac{1}{\sqrt{2}}(2 - \sqrt{2})^{-1}.$$

Therefore,

$$a_{2n} = \frac{1}{\sqrt{2}} \left[ (2 + \sqrt{2})^{n-1} - (2 - \sqrt{2})^{n-1} \right],$$

for  $n \geq 1$ .