math 222

SOLUTIONS TO FINAL EXAMINATION

1. The following message was encrypted using a linear code.

NYT ENDY

The encoding function used was:

$$E(x) \equiv 15x - 15 \text{ (mod 26)}.$$

Decipher the message.

SOLUTION: Since $7 \cdot 15 \equiv 105 \equiv 1$ (mod 26), then 15^{-1} modulo 26 is 7, so that

$$7 \cdot E(x) \equiv 7 \cdot 15(x-1) \equiv x - 1 \text{ (mod26)},$$

and therefore,

$$x \equiv 1 + 7 \cdot E(x) \text{ (mod26)},$$

so that the decoding function is

$$D(x) \equiv 1 + 7 \cdot x \text{ (mod26)}.$$

Now, using the correspondence between the letters of the alphabet and the nonnegative integers given below,

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25

we have the following modulo 26:

$$D(N) \equiv D(13) \equiv 1 + 7 \cdot 13 \equiv 1 + 91 \equiv 92 \equiv 14 \equiv O$$

$$D(Y) \equiv D(24) \equiv 1 + 7 \cdot 24 \equiv 1 + 168 \equiv 169 \equiv 13 \equiv N$$

$$D(T) \equiv D(19) \equiv 1 + 7 \cdot 19 \equiv 1 + 133 \equiv 134 \equiv 4 \equiv E$$

$$D(E) \equiv D(4) \equiv 1 + 7 \cdot 4 \equiv 1 + 28 \equiv 29 \equiv 3 \equiv D$$

$$D(N) \equiv D(13) \equiv 1 + 7 \cdot 13 \equiv 1 + 91 \equiv 92 \equiv 14 \equiv O$$

$$D(D) \equiv D(3) \equiv 1 + 7 \cdot 3 \equiv 1 + 21 \equiv 22 \equiv W$$

$$D(Y) \equiv D(24) \equiv 1 + 7 \cdot 24 \equiv 1 + 168 \equiv 169 \equiv 13 \equiv N$$

The decoded message is

ONE DOWN

2. Baskerhound had herded Evangeline and her friends into a room containing four caskets. Baskerhound stated that sixty minutes after he leaves the room, three of the four caskets will disintegrate and release a poison gas. The other casket contains the key to the door. They must find the key before the hour is up to escape. He gave them a clue: The key is in casket number $3^n + 1$ where $n > 1000$, and it is also in casket number $5^m - 1$ where $m > 1000$. But there is only **one** key.

   You count the caskets as follows: Casket 1 was 1, Casket 2 was 2, and so on, until Casket 4 which was 4. Then the kidnapper started counting backwards: Casket 3 was 5, Casket 2 was 6, and Casket 1 was 7. Then the count reversed once more, and Casket 2 was 8, Casket 3 was 9, etc.

   Find the casket which contains the key.

   SOLUTION: Evangeline started counting the caskets as shown in the figure below:

1	2	3	4
7	6	5	
	8	9	10
13	12	11	
	14	15	16
19	18	17	
	20	21	22
25	24	23	
	26	27	28
⋮	⋮	⋮	⋮

   She quickly noticed that

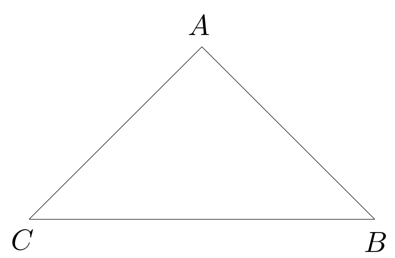
   (1) The numbers labeling casket #1 were all congruent to 1 modulo 6.
   (2) The numbers labeling casket #2 were all congruent to 0 or 2 modulo 6.
   (3) The numbers labeling casket #3 were all congruent to 3 or 5 modulo 6.
   (4) The numbers labeling casket #4 were all congruent to 4 modulo 6.

   Baskerhound had stated that the key was in a casket labeled $3^n + 1$ with $n > 1,000$, and also in a casket labeled $5^m - 1$ with $m > 1,000$, so Evangeline began computing: modulo 6

   (a) For n even, say $n = 2k$, we have $3^{2k} + 1 \equiv 3 + 1 \equiv 4 \pmod 6$.
   (b) For n odd, say $n = 2k + 1$, we have $3^{2k+1} + 1 \equiv 3 + 1 \equiv 4 \pmod 6$.
   (c) For m even, say $m = 2k$, we have $5^{2k} - 1 \equiv 1 - 1 \equiv 0 \pmod 6$.
   (d) For m odd, say $m = 2k + 1$, we have $5^{2k+1} - 1 \equiv 5 - 1 \equiv 4 \pmod 6$.

   She immediately deduced that the key was in casket #4.

3. A bug starts at vertex A of the triangle below and each minute travels to an adjacent vertex. There is a spider web on vertex C; if the bug moves there it is stuck. Let a_n be the number of different ways the bug can travel from vertex A to vertex C after n minutes. Set up a recurrence relation for a_n (remember to include the initial conition(s)). Solve the recurrence relation and find a_{20}.



Solution:

- For $n = 1$, the bug has to go directly from A to C in order to get there in 1 minute, therefore $a_1 = 1$.
- For $n \geq 2$, the bug must first go to B, and then can move to C in b_{n-1} ways, so that $a_n = b_{n-1}$. However, by symmetry $a_n = b_n$ for all $n \geq 1$.

Therefore, a_n satifies the discrete initial value problem

$$a_n = a_{n-1}, \quad n \geq 2$$

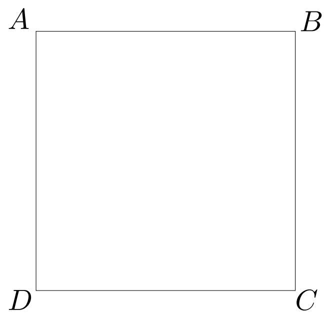
$$a_1 = 1.$$

If we iterate the recurrence relation, we have

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 = 1 \\ a_3 &= a_2 = 1 \\ &\vdots \\ a_n &= 1 \\ &\vdots \end{aligned}$$

and it looks like $a_n = 1$ for all $n \geq 1$. An easy induction argument verifies this, and therefore $a_{20} = 1$.

4. A bug starts at vertex A of the square below and each minute travels to an adjacent vertex. There is a spider web on vertex C; if the bug moves there it is stuck. Let a_n be the number of different ways the bug can travel from vertex A to vertex C after n minutes. Set up a recurrence relation for a_n (remember to include the initial condition(s)). Solve the recurrence relation to find a_{20}.



Solution:

- For $n = 1$, we have $a_1 = 0$ since the shortest path between A and C has length 2.
- For $n = 2$, we have two paths from A to C of length 2, one going through B and the other going through D, so that $a_2 = 2$.
- For $n = 3$, if the bug goes to B first, then it must return to A since otherwise it arrives at C too early, and then there is no way to reach C in one minute. Similarly, if the bug goes to D first, then it must return to A, and again there is no way to reach C in one minute. Therefore, $a_3 = 0$.
- For $n \geq 4$, if the bug goes to B first, it must return to A, and then there are a_{n-2} ways to get to C in n minutes. Similarly, if the bug goes to D first. it must return to A, and then there are a_{n-2} ways to get to C in n minutes. Thus, $a_n = a_{n-2} + a_{n-2} = 2a_{n-2}$ for $n \geq 4$.

Therefore, a_n satisfies the discrete initial value problem

$$a_n = 2a_{n-2}, \quad n \geq 3,$$

$$a_1 = 0,$$

$$a_2 = 2.$$

Iterating, we find the solution

$$a_{2n} = 2^n$$

and

$$a_{2n-1} = 0$$

for $n \geq 1$. Therefore, $a_{20} = 2^{10} = 1024$.

5. Use strong induction to show the Fibonacci numbers can be written in closed form:

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)$$

for $n \geq 1$, where

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

SOLUTION: We use strong induction on n.

Base Case: For $n = 1$, we have

$$\frac{1}{\sqrt{5}}\left(\alpha^1 - \beta^1\right) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}\right) = 1 = F_1,$$

and the result is true for $n = 1$.

Inductive Step: Assume the result is true for all integers k with $1 \leq k \leq n$. Since both α and β satisfy the quadratic $x^2 = x + 1$, then

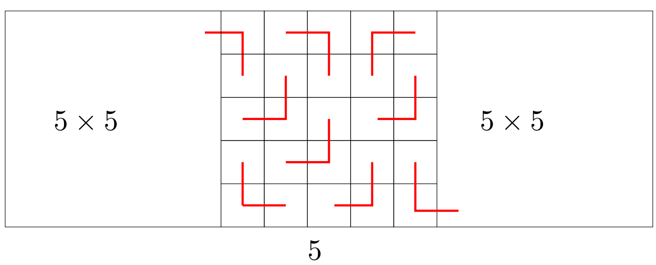
$$\begin{aligned}
\frac{1}{\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right) &= \frac{1}{\sqrt{5}}\left(\alpha^{n-1}\alpha^2 - \beta^{n-1}\beta^2\right) \\
&= \frac{1}{\sqrt{5}}\left(\alpha^{n-1}(\alpha+1) - \beta^{n-1}(\beta+1)\right) \\
&= \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right) + \frac{1}{\sqrt{5}}\left(\alpha^{n-1} - \beta^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right) + \frac{1}{\sqrt{5}}\left(\alpha^{n-1} - \beta^{n-1}\right),
\end{aligned}$$

and from the induction hypotheses, we have

$$\frac{1}{\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right) = F_n + F_{n-1} = F_{n+1}.$$

6. Show that a 5×15 board can be tiled with right trominoes. Use the fact that some deficient 5×5 boards can be tiled with right trominoes by sectioning a 5×5 board on each end of the 5×15 board.

SOLUTION: A tiling of the 5×15 board is shown below.



Where the outer 5×5 deficient boards can be tiled according to Lecture 16, Example 2.

7. After a recent sucessful tiling problem, Professor Scarlett has attempted an inductive proof of the following statement for $n \geq 6$ and $n \equiv\ 0, 2 \pmod 3$.

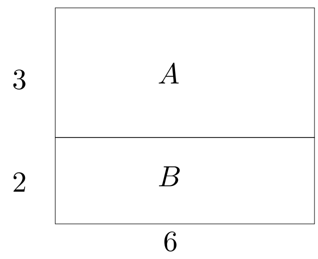
   P_n: A $5 \times n$ board can be tiled given right trominoes and a single monomial.

   Unfortunately Professor Scarlett forgot to include the base case(s). Help Professor Scarlett complete the proof by completing the base case(s). You must use the smallest number of base cases necessary to complete the proof.

   SOLUTION:

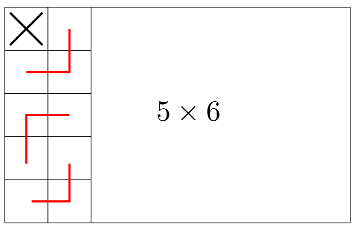
   Base Case(s): For the base cases we will show that P_6 is true, and P_6 true implies that P_8 is true, and finally that P_9 and P_{11} are true.

   P_6 : We partition the 5×6 board as follows:



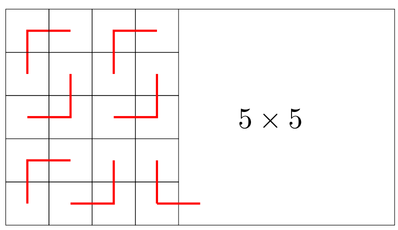
   Since $3 \equiv 6 \equiv 0 \pmod 3$ and $6 \equiv 2 \equiv 0 \pmod 2$, section A and B can be tiled by Lecture 16, Proposition 1.

   P_8 : We partition the 5×8 board as follows:

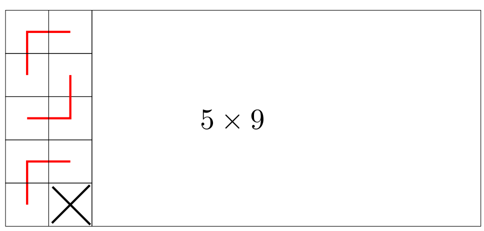


   The left side is a 5×2 board and is tiled as shown using trominoes and the monomial, while the 5×6 board on the right can be tiled by P_6 above.

   P_9 : We partition the 5×9 board into a 5×4 board and a 5×5 board as shown. We tile the 5×4 board as shown, and then tile the deficient 5×5 board remaining as in Lecture 16, Example 2.



   P_{11} : We partition the 5×11 board into a 5×2 board and a 5×9 board as follows:



   We tile the 5×2 using trominoes and the monomial as shown, while the 5×9 board on the right can be tiled by P_9 above.

Inductive Step: The integers $n \geq 6$ with $n \equiv 0 \pmod 3$ are

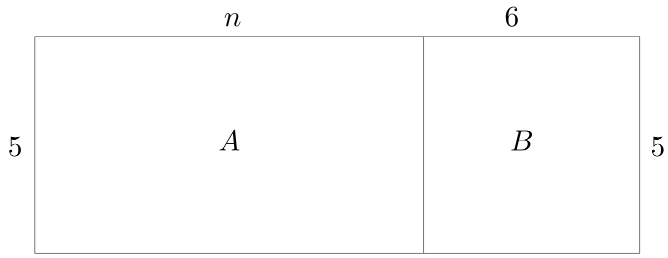
$$6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 24,\ 27,\ 30,\ \ldots,$$

while the integers $n \geq 6$ with $n \equiv 2 \pmod 3$ are

$$8,\ 11,\ 14,\ 17,\ 20,\ 23,\ 26,\ 29,\ 32,\ \ldots.$$

If we know that P_6, P_8, P_9, and P_{11} are true, and if we can show that whenever P_n is true, this implies that P_{n+6} is true, then we can conclude by induction that P_n is true for all $n \geq 6$ with $n \equiv 0, 2 \pmod 3$.

Suppose that $n \geq 6$ and $n \equiv 0, 2 \pmod 3$ and P_n is true, in order to show P_{n+6} is true, we partition a $5 \times (n+6)$ board as follows:



Since P_n is true, then we can tile the board A using right trominoes and a single monomial, and the board B can be tiled with trominoes by P_6. Therefore, if P_n is true, then P_{n+6} is true.

Note: The minimum number of bases cases needed to complete the proof by induction is four:

$$P_6, \quad P_8, \quad P_9, \quad \text{and} \quad P_{11}.$$

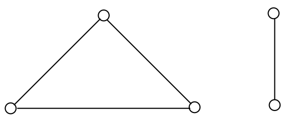
8. In every tree, there is one more vertex than the number of edges, this is called the tree formula:

$$V = E + 1.$$

Construct a graph where the tree formula is true but the graph is not a tree.

SOLUTION: Since a graph G is a tree if and only if G is connected and satisfies the tree formula, we need to find a graph with V vertices and E edges such that $V = E + 1$, but which is not connected, as in the figure below.

G :



Here $V = 5$ and $E = 4$, so that

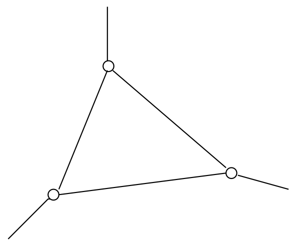
$$V = 5 = 4 + 1 = E + 1,$$

but the graph is not a tree since it is not connected.

9. Kids and Chocolate bars. What are the possible values of $n > 6$ such that n children can equally share 6 identical chocolate bars, with the restriction that no bar can be cut into more than two pieces.

SOLUTION: We represent the problem using a graph G with n vertices representing the children, and an edge between two vertices if and only if the corresponding children share a chocolate bar.

Now note that this graph cannot contain a cycle, since this would imply that the children representing the vertices in the cycle would each receive at least one full chocolate bar.



Therefore, the graph is a forest, that is, a disjoint collection of trees. Suppose that there are $k > 1$ trees in the forest, one with e edges and one with f edges, then since each of the children get an equal amount of chocolate, we have

$$\frac{e}{e+1} = \frac{f}{f+1},$$

and this implies that $e = f$. Thus, each of the trees in the forest has the same number of edges, say e.

If k is the number of trees, then we must have

$$k \cdot e = 6,$$

and k must be a divisor of 6, that is, $k = 1, 2, 3, 6$.

- If $k = 1$, then $e = 6$, and there is one tree with $e = 6$ edges, and in this case

$$n = e + 1 = 6 + 1 = 7.$$

- If $k = 2$, then $e = 3$, and there are two trees each with $e = 3$ edges, so

$$n = 2(e + 1) = 2 \cdot 4 = 8.$$

- If $k = 3$, then $e = 2$, and there are three trees each with $e = 2$ edges, so

$$n = 3(e + 1) = 3 \cdot 3 = 9.$$

- If $k = 6$, then $e = 1$, and there are six trees each with $e = 1$ edge, so

$$n = 6(e + 1) = 6 \cdot 2 = 12.$$

Therefore, the number of children could be $n = 7, 8, 9, 12$.

10. If 22 pigeons each choose one of 5 pigeonholes to fly into, then how many of the statements below must be true:

(1) At least 5 pigeons flew into one hole.

(2) At most 17 pigeons flew into some combination of 4 pigeonholes.

(3) At least 10 pigeons flew into some combination of 2 pigeonholes.

(4) At most 12 pigeons flew into some combination of 3 pigeonholes.

(5) At least 14 pigeons flew into some combination of 3 pigeonholes.

(6) At most 8 pigeons flew into some combination of 2 pigeonholes.

(7) At least 18 pigeons flew into some combination of 4 pigeonholes.

(8) At most 4 pigeons flew into 1 hole.

SOLUTION: Let a_i be the number of pigeons that flew into pigeonhole i, for $i = 1, 2, \ldots, 5$.

(1) Suppose that no more than 4 pigeons flew into each of the 5 pigeonholes, then

$$a_i \leq 4$$

for $i = 1, 2, \ldots, 5$, and we would have

$$22 = \sum_{i=1}^{5} a_i \leq \sum_{i=1}^{5} 4 = 5 \cdot 4 = 20,$$

which is a contradiction. Therefore, the statement is true.

(2) From (1), at least one pigeon hole contains 5 or more pigeons, and therefore at most $22 - 5 = 17$ pigeons flew into some combination of 4 pigeonholes. Therefore, the statement is true.

(3) From (1), at least one pigeonhole contains 5 or more pigeons, say $a_5 \geq 5$. If one of the other pigeonholes has at least 5 pigeons, then we are done, since these two pigeon holes contain at least 10 pigeons.
If not, then $a_i \leq 4$ for $i = 1, 2, 3, 4$, and

$$22 - a_5 = \sum_{i=1}^{4} a_i \leq \sum_{i=1}^{4} 4 = 16,$$

and so in fact $a_5 \geq 22 - 16 = 6$. If one of $a_i = 4$ for some $1 \leq i \leq 4$, then again, we are done.
If not, then $a_i \leq 3$ for $i = 1, 2, 3, 4$, and

$$22 - a_5 = \sum_{i=1}^{4} a_i \leq \sum_{i=1}^{4} 3 = 12,$$

and so in fact $a_5 \geq 22 - 12 = 10$, and we are done. Therefore, the statement is true.

(4) From (3) at least 10 pigeons flew into some combination of 2 pigeonholes, and therefore at most $22 - 10 = 12$ pigeons flew into some combination of 3 pigeonholes. Therefore, the statement is true.

(5) From (3), at least 10 pigeons flew into some combination of 2 pigeonholes, say $a_4 + a_5 \geq 10$. If one of the other pigeonholes has at least 4 pigeons, then we are done, since these 3 pigeon holes contain at least 14 pigeons.
If not, then $a_i \leq 3$ for $i = 1, 2, 3$, and

$$22 - a_4 - a_5 = \sum_{i=1}^{3} a_i \leq \sum_{i=1}^{3} 3 = 9,$$

and so in fact $a_4 + a_5 \geq 22 - 9 = 13$. If one of $a_i = 3$ for $i = 1, 2, 3$, then again, we are done.
If not, then $a_i \leq 2$ for $i = 1, 2, 3$, and

$$22 - a_4 - a_5 = \sum_{i=1}^{3} a_i \leq \sum_{i=1}^{3} 2 = 6,$$

and so in fact $a_4 + a_5 \geq 22 - 6 = 16$, and we are done. Therefore, the statement is true.

(6) From (5), at least 14 pigeons flew into some combination of 3 pigeonholes, and therefore at most $22 - 14 = 8$ pigeons fllew into some combination of 2 pigeonholes.

(7) From (5), at least 14 pigeons flew into some combination of 3 pigeonholes, say $a_3 + a_4 + a_5 \geq 14$. If one of the other pigeonholes has at least 4 pigeons, then we are done, since these 4 pigeonholes contain at least 18 pigeons.
If not, then $a_i \leq 3$ for $i = 1, 2$, and

$$22 - a_3 - a_4 - a_5 = \sum_{i=1}^{2} a_i \leq \sum_{i=1}^{2} 3 = 6,$$

and so in fact $a_3 + a_4 + a_5 \geq 22 - 6 = 16$. If one of $a_i \geq 2$ for $i = 1, 2$, then again, we are done.
If not, then $a_i \leq 1$ for $i = 1, 2$, then in fact

$$a_3 + a_4 + a_5 = 22 - a_1 - a_2 \geq 22 - 2 = 20,$$

and we are done. Therefore, the statement is true.

(8) From (7), at least 18 pigeons flew into some combination of 4 pigeonholes, and therefore at most $22 - 18 = 4$ pigeons flew in the remaining pigeonhole.

Alternative solution: First note that:

$$\begin{aligned} (1) &\iff (2) \\ (3) &\iff (4) \\ (5) &\iff (6) \\ (7) &\iff (8). \end{aligned}$$

Next note that: If at most p pigeons fly into h pigeonholes, where $p \geq h$, then there are $h-1$ pigeonholes with at most $p - \left\lceil \frac{p}{h} \right\rceil$ pigeons. Therefore,

$$(2) \implies (4) \implies (6) \implies (8),$$

and thus,

$$\begin{array}{ccccccc} (1) & & (3) & & (5) & & (7) \\ \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\ (2) & \implies & (4) & \implies & (6) & \implies & (8) \end{array}$$

and since (1) is true by the pigeonhole principle, then all 8 of the statements are true.