MATH 222

Assignment#4

Due: 5pm on the date stated in the course outline.
Hand in to the assignment box on the 3rd floor of CAB.

1. Let a_n denote the number of interior regions of a convex polygon with n sides, divided by all its diagonals, if no three diagonals pass through a common point inside the polygon. Use the theorem below to show that:

   $$a_n = \binom{n-1}{2} + \binom{n}{4}$$

   for $n \geq 4$.

   Theorem: Consider a convex region of the plane which is crossed by l lines with p interior points of intersection. No three of the lines pass through a common point inside the region. The number of disjoint regions created is:

   $$1 + l + p$$

Solution: There is one interior point of intersection for every choice of four points on the convex polygon, therefore $p = \binom{n}{4}$. There is one line for every two points on the convex polygon but n of these lines are not diagonals, therefore $l = \binom{n}{2} - n$. Using the theorem we have:

$$\begin{aligned}
a_n &= 1 + l + p \\
&= 1 + \binom{n}{2} - n + \binom{n}{4} \\
&= \frac{n(n-1)}{2} - n + 1 + \binom{n}{4} \\
&= \frac{n(n-1)}{2} - (n-1) + \binom{n}{4} \\
&= (n-1)\left(\frac{n}{2} - 1\right) + \binom{n}{4} \\
&= (n-1)\left(\frac{n-2}{2}\right) + \binom{n}{4} \\
&= \binom{n-1}{2} + \binom{n}{4}
\end{aligned}$$

2. Use mathematical induction to prove that the sum of the first n square numbers is equal to $\frac{n(n+1)(2n+1)}{6}$. That is, prove:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:

Base case: $n=1$: $\sum_{i=1}^{1} i^2 = 1^2 = 1$ and $\frac{1(1+1)(2+1)}{6} = 1$, so the base case holds. ✓

Show: $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \Longrightarrow \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 \overset{\boxed{\text{By induction assumption}}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$ ✓

3. Conjecture a formula for the sum of the first n Fibonacci numbers with odd indices and prove your formula works by using mathematical induction. That is, find and prove a formula for

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1}$$

Note:

$$F_1 = 1,$$
$$F_2 = 1,$$
$$F_n = F_{n-1} + F_{n-2}.$$

Solution:

Recall, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$

The first few terms gives us:

$n = 1$: $1 = 1$
$n = 2$: $1 + 2 = 3$
$n = 3$: $1 + 2 + 5 = 8$
$n = 4$: $1 + 2 + 5 + 13 = 21$
$n = 5$: $1 + 2 + 5 + 13 + 34 = 55$

Perhaps the sum of the first n Fibonacci numbers with odd indices is F_{2n} .

✓

Base case: $n = 1$

$F_1 = 1 = F_{2\cdot 1} = F_2$, so the base case holds.

Inductive Step:

Show: $F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}$

$\Rightarrow$ $F_1 + F_3 + F_5 + \cdots + F_{2k-1} + F_{2k+1} = F_{2(k+1)}$

$$F_1 + F_3 + F_5 + \cdots + F_{2k-1} + F_{2k+1} \underset{\substack{\uparrow \\ \boxed{\text{By induction assumption}}}}{=} F_{2k} + F_{2k+1} \underset{\substack{\uparrow \\ \boxed{\text{By Fibonacci recurrence}}}}{=} F_{2k+2} = F_{2(k+1)}$$

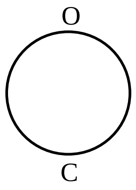
✓

4. An equal number of open and closed gas stations are distributed an equal distance apart around a ring road of a city. Say n stations are open and n stations are closed. Now suppose it takes n gallons of gas to travel around the city. If we start with an empty tank and can only add 1 gallon at a time at any open station, show that it possible to find a starting position to travel around the city clockwise for all $n \geq 1$?

Solution:

Base case: $n = 1$

When the is only one open and one closed gas station we can start at the open gas station:

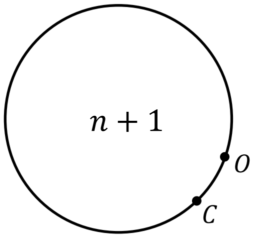


✓

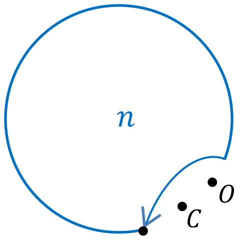
Inductive Step:

Show: if it possible to find a starting position to travel around the city with n open and n closed gas stations then it possible to find a starting position to travel around the city with $n + 1$ open and $n + 1$ closed gas stations.

Consider the ring road with $n + 1$ open and $n + 1$ closed gas stations



Search for an open gas station O which is followed by a closed gas station C. Notice that during travel around the city if one adds a gallon at O one can always travel one gas station past C. Noting that reduces the problem to finding a starting position among n open and n closed gas stations:

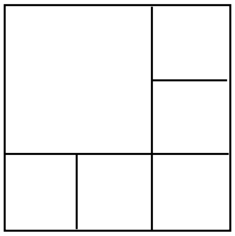


and by our assumption there is a starting position.

✓

5. Prove that a square can be dissected into n smaller squares for any positive integer $n \geq 6$. (The smaller squares do not all have to be the same size)

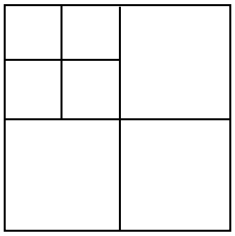
   For example a square can be dissected into 6 smaller squares:

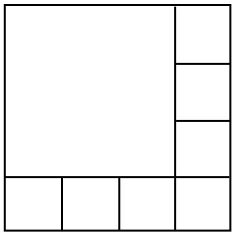


Solution: Let P_n be the statement for $n \geq 6$.

Base Cases:

The example given shows P_6 is true. The following two figures are dissections into 7 and 8 squares respectively:



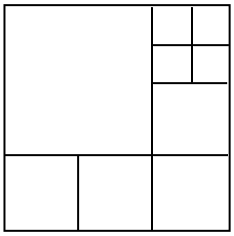


Therefore P_7 and P_8 are true.

Inductive Step:

Show: $P_n \Rightarrow P_{n+3}$.

Notice that if a square can be dissected into n smaller squares, then it can be dissected into $n + 3$ smaller squares. This is done by taking one of the existing squares and dissecting it into four squares of equal size. For example, using the dissection of 6 squares, the following is a dissection into 9 squares:



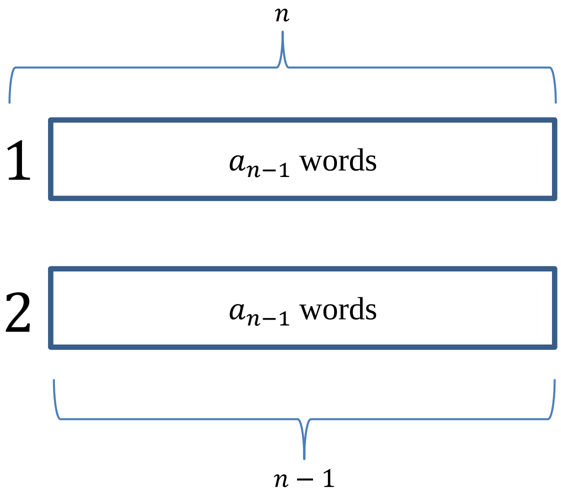
Therefore if P_n is true then P_{n+3} is also true.

6. Let a_n be the number of words of length n with digits from the set $\{0,1,2\}$ with no consecutive 0's.

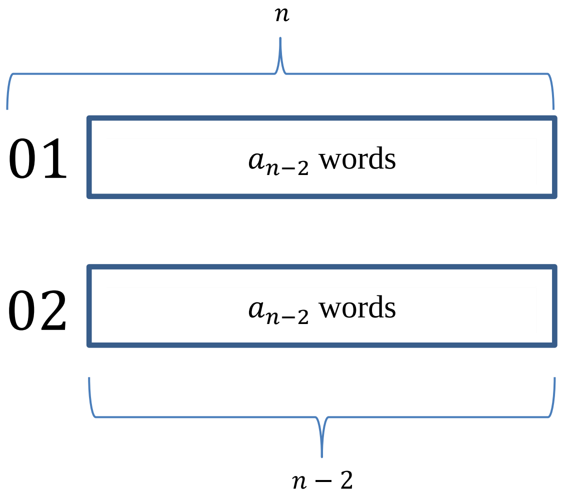
   a) Show that: $a_1 = 3$, $a_2 = 8$, $a_n = 2a_{n-1} + 2a_{n-2}$ is a recurrence relation for a_n.

Solution: When $n = 1$ the words 0, 1, and 2 do not have consecutive 0's, therefore $a_1 = 3$. For $n = 2$ there are 8 words that do not have consecutive 0's: 01, 02, 10, 11, 12, 20, 21, 22 therefore $a_2 = 8$. To find a recurrence relation for a_n we look at 2 cases.

CASE 1: The first digit is not a zero. Then the first digit is a 1 or a 2 in either case there are a_{n-1} ways to complete the word with no consecutive 0's.



CASE 2: The first digit is a zero. Then the second digit must be a 1 or 2 in either case there are a_{n-2} ways to complete the word with no consecutive 0's.



Therefore: $a_1 = 3$, $a_2 = 8$, $a_n = 2a_{n-1} + 2a_{n-2}$ is a recurrence relation for a_n.

b) Use strong induction to show that:

$$a_n = \frac{\left(1+\sqrt{3}\right)^{n+2} - \left(1-\sqrt{3}\right)^{n+2}}{4\sqrt{3}}$$

is a solution to the recurrence relation from part a).

Solution: Let P_n be the statement for $n \geq 1$. Base Cases:

$$n = 1 \Rightarrow \frac{\left(1+\sqrt{3}\right)^{3} - \left(1-\sqrt{3}\right)^{3}}{4\sqrt{3}} = 3 = a_1$$

$$n = 2 \Rightarrow \frac{\left(1+\sqrt{3}\right)^{3} - \left(1-\sqrt{3}\right)^{3}}{4\sqrt{3}} = 3 = a_2$$

Therefore P_1 and P_2 are true.

Inductive Step: Show: P_{n-1} & $P_{n-2} \Rightarrow P_n$.

$$a_n = 2a_{n-1} + 2a_{n-2}$$

$$= 2\frac{\left(1+\sqrt{3}\right)^{n+1} - \left(1-\sqrt{3}\right)^{n+1}}{4\sqrt{3}} + 2\frac{\left(1+\sqrt{3}\right)^{n} - \left(1-\sqrt{3}\right)^{n}}{4\sqrt{3}} \quad \text{(by } P_{n-1} \text{ \& } P_{n-2}\text{)}$$

$$= \frac{2\left(1+\sqrt{3}\right)^{n+1} + 2\left(1+\sqrt{3}\right)^{n} - 2\left(1-\sqrt{3}\right)^{n+1} - 2\left(1-\sqrt{3}\right)^{n}}{4\sqrt{3}}$$

$$= \frac{2\left(1+\sqrt{3}\right)^{n+1} + 2\left(1+\sqrt{3}\right)^{n} - 2\left(1-\sqrt{3}\right)^{n+1} - 2\left(1-\sqrt{3}\right)^{n}}{4\sqrt{3}}$$

$$= \frac{\left(1+\sqrt{3}\right)^{n}\left(2\left(1+\sqrt{3}\right)+2\right) - \left(1-\sqrt{3}\right)^{n}\left(2\left(1-\sqrt{3}\right)+2\right)}{4\sqrt{3}}$$

$$= \frac{\left(1+\sqrt{3}\right)^{n}\left(1+2\sqrt{3}+3\right) - \left(1-\sqrt{3}\right)^{n}\left(1-2\sqrt{3}+3\right)}{4\sqrt{3}}$$

$$= \frac{\left(1+\sqrt{3}\right)^{n}\left(1+\sqrt{3}\right)^{2} - \left(1-\sqrt{3}\right)^{n}\left(1-\sqrt{3}\right)^{2}}{4\sqrt{3}}$$

$$= \frac{\left(1+\sqrt{3}\right)^{n+2} - \left(1-\sqrt{3}\right)^{n+2}}{4\sqrt{3}}$$

Therefore P_n is true.

7. Prove Cassini's Identity:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

For $n \geq 1$.

Note:

$$F_0 = 0,$$
$$F_1 = 1,$$
$$F_n = F_{n-1} + F_{n-2}.$$

Solution: Let P_n be the statement for $n \geq 1$. Base Case:

$$n = 1 \Rightarrow F_2F_0 - F_1^2 = 1 \cdot 0 - 1^2 = -1 = (-1)^1$$

Therefore P_1 is true.

Inductive Step: Show: $P_{n-1} \Rightarrow P_n$.

$$F_{n+1}F_{n-1} - F_n^2$$

$$= (F_n + F_{n-1})(F_{n-1}) - (F_{n-1} + F_{n-2})F_n$$

$$= F_nF_{n-1} + F_{n-1}^2 - F_{n-1}F_n - F_{n-2}F_n$$

$$= F_{n-1}^2 - F_{n-2}F_n$$

$$= -(F_nF_{n-2} - F_{n-1}^2)$$

$$= -(-1)^{n-1} \qquad \text{(by } P_{n-1}\text{)}$$

$$= (-1)(-1)^{n-1}$$

$$= (-1)^{n-1+1}$$

$$= (-1)^n$$

Therefore P_n is true.

Another Solution: For readers who have a linear algebra background.

First notice that:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} F_4 & F_3 \\ F_3 & F_2 \end{bmatrix}$$

$$\vdots$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

Now take the determinant of both sides:

$$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^n = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix}$$

$$\Rightarrow (-1)^n = F_{n+1}F_{n-1} - F_n^2$$

8. The objective of this question is to prove the following theorem.

   For all natural numbers $n \neq 5$:

   “a deficient $n \times n$ board can be tiled with right trominoes” $\Longleftrightarrow$ $n \not\equiv 0 \pmod 3$.

   Recall: when $n \equiv 1 \pmod 3$ every deficient $n \times n$ board can be tiled with trominoes.
   Finish proving the theorem by completing parts a) and b).

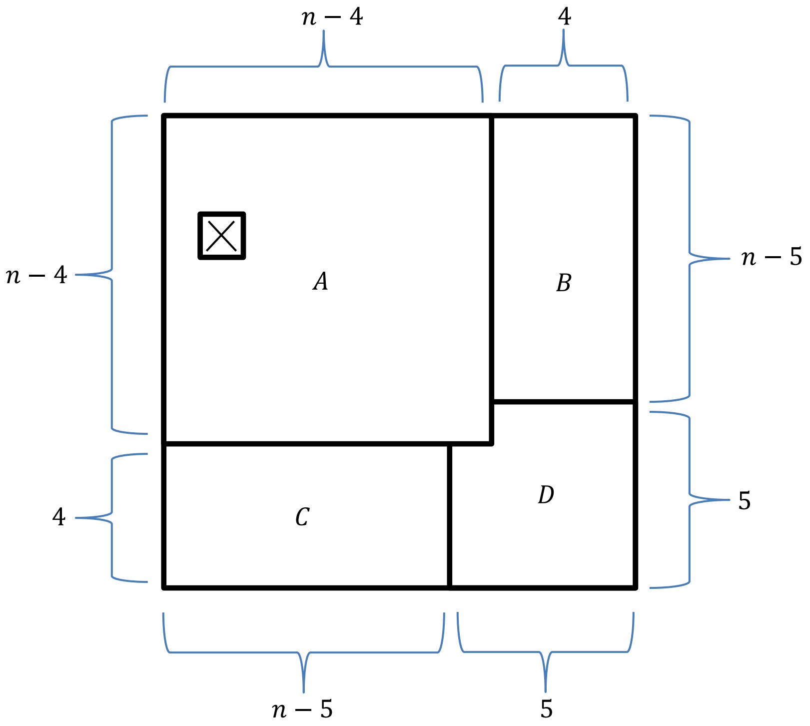
   a) Show that if $n \equiv 0 \pmod 3$ then a deficient $n \times n$ board cannot be tiled with trominoes.
   b) Show that if $n \equiv 2 \pmod 3$ and $n \neq 5$ then a deficient $n \times n$ board can be tiled with trominoes.

Solution: a) Note: a deficient $n \times n$ board has an area of: $n^2 - 1$. In order to tile any board with trominoes its area must be divisible by 3 but if $n \equiv 0 \bmod 3$ then:

$$n^2 - 1 \equiv 0^2 - 1 \equiv 1 \not\equiv 0 \bmod 3.$$

Therefore, if $n \equiv 0 \bmod 3$ a deficient $n \times n$ board cannot be tiled with trominoes.

b) Note that the special cases of $n = 2, 8$ have both been shown in example 1 of lecture 16, so we can safely assume that $n \geq 11$. Let $n \equiv 2 \pmod 3$ for $n \geq 11$ and section off a deficient $n \times n$ board as follows:



Now by symmetry we only need to consider the case when the missing square is in section A. Each section can be tiled with trominoes for the following reasons:

- $n - 4 \equiv 2 - 4 \equiv 2 - 1 \equiv 1 \bmod 3 \quad \xRightarrow{Lecture\ 16\ \ Prop.2} \quad$ Section A can be tiled with trominoes.
- $n - 5 \equiv 2 - 5 \equiv 2 + 1 \equiv 0 \bmod 3$, $4 \equiv 0 \bmod 2 \quad \xRightarrow{Lecture\ 16\ \ Prop.1} \quad$ Sections B and C can both be tiled with trominoes.
- Section D is a deficient 5×5 board missing its top left corner. $\quad \xRightarrow{Lecture\ 16\ \ Ex2} \quad$ Section D can be tiled with trominoes.

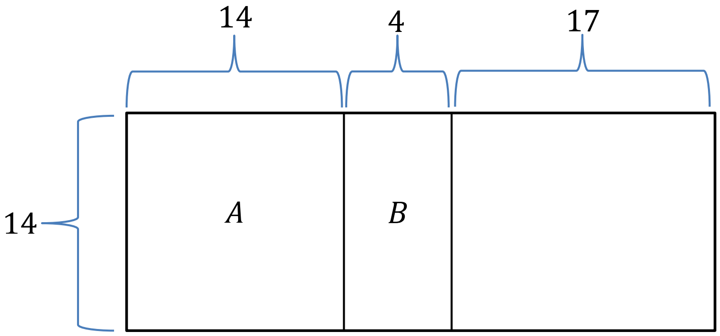
Therefore, if $n \equiv 2 \pmod 3$ and $n \neq 5$ a deficient $n \times n$ board can be tiled with trominoes.

9. For all $n \geq 0$ show that any deficient $14 \cdot 2^n \times 35 \cdot 2^n$ board can be tiled with right trominoes.

Solution:

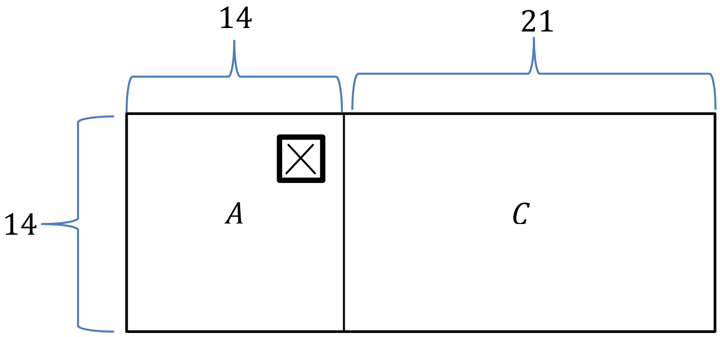
Base case: $n = 0$

In this case we have a 14×35 board and start by breaking it into the following sections:



By symmetry we only need to consider the case when the missing square is in section A or B.

CASE 1: the missing square is in section A

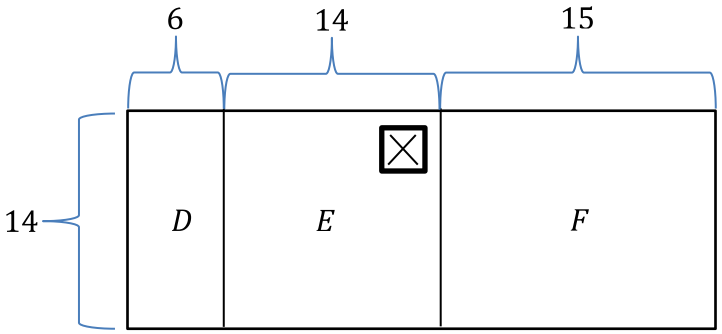


Now:

- $14 \equiv 2 \bmod 3 \xRightarrow{problem\ 7}$ Section A can be tiled with trominoes.
- $\begin{matrix} 21 \equiv 0 \bmod 3 \\ 14 \equiv 0 \bmod 2 \end{matrix} \xRightarrow{Lecture\ 16\ \ Prop.1}$ Section C can be tiled with trominoes.

CASE 2: the missing square is in section B

Place a 14×14 block around section B as follows:

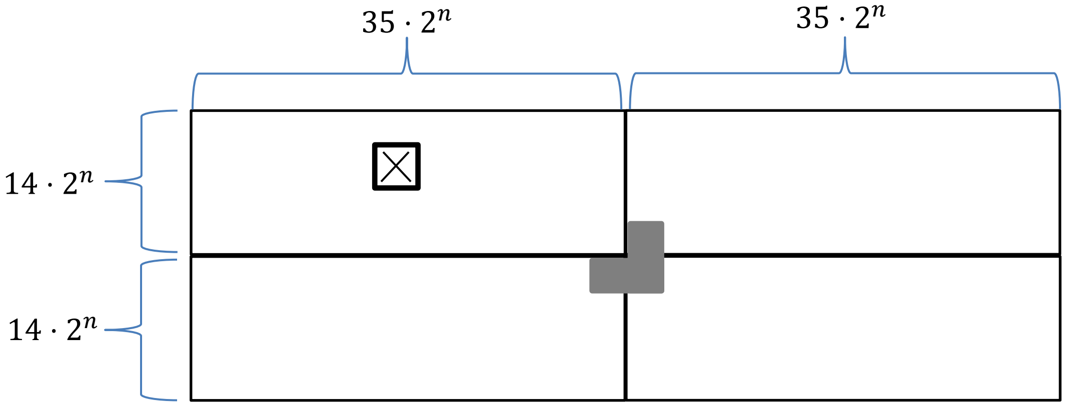


Now,

- $14 \equiv 1 \bmod 3 \quad \xRightarrow{\textit{problem 7}} \quad$ Section E can be tiled with trominoes.
- $6 \equiv 15 \equiv 0 \bmod 3$, $14 \equiv 0 \bmod 2 \quad \xRightarrow{\textit{Lecture 16 Prop.1}} \quad$ Sections D and F can be tiled with trominoes.

<u>Inductive step:</u> Show: if any deficient $14 \cdot 2^n \times 35 \cdot 2^n$ board can be tiled with trominoes then any deficient $14 \cdot 2^{n+1} \times 35 \cdot 2^{n+1}$ board can be tiled with trominoes (for $n \geq 0$).

In a deficient $14 \cdot 2^{n+1} \times 35 \cdot 2^{n+1}$ board by symmetry we only need to consider the case when the missing square is in the top left:



By placing a tromino in the center as shown we create four deficient $14 \cdot 2^n \times 35 \cdot 2^n$ boards which can be tiled by induction.

10. The principle of double induction is stated as follows.

If $P(m,n)$ is a statement about the integers $m \geq a$ and $n \geq b$ such that

1. $P(a,b)$ is true,
2. For all $m \geq a$, if $P(m,b)$ is true, then $P(m+1,b)$ is true,
3. For all $n \geq b$, if $P(m,n)$ is true for all $m \geq a$, then $P(m,n+1)$ is true for all $m \geq a$,

then $P(m,n)$ is true for all $m \geq a$ and $n \geq b$.

Use the principle of double induction to show that for any integers $m \geq 1$ and $n \geq 1$ we have:

$$(m+1)^n > mn$$

Solution:

Let $P(m,n)$ be the statement that $(m+1)^n > mn$ for $m,n \geq 1$.

<u>Step 1</u>: Show: $P(1,1)$ is true.

$$(m+1)^n = (1+1)^1 = 2 > 1 = 1 \cdot 1 = mn$$

<u>Step 2</u>: Show $P(m,1) \Rightarrow P(m+1,1)$ for $m \geq 1$.

$$(m+1)^1 = m+1 > m = m \cdot 1$$

Therefore $P(m+1,1)$ is true.

<u>Step 3</u>: Show $P(m,n) \Rightarrow P(m,n+1)$ for $m,n \geq 1$.

$$(m+1)^{n+1} = (m+1)^n(m+1) > mn(m+1) = mn + m^2n > mn + m = m(n+1)$$

Therefore $P(m,n+1)$ is true.

Bonus.

A game is played with a deck of k cards numbered from 1 to k. They are shuffled thoroughly and the top card is turned over. If it is number 1, the game is won. If it is number i where $2 \leq i \leq k$, then it is inserted into the deck so that it is the i-th card from the top. Then the new top card is turned over and the same process is applied. Can this game be won eventually, regardless of how the cards are stacked?

Solution. The game can always be won by induction.

Base case: $(k = 1)$

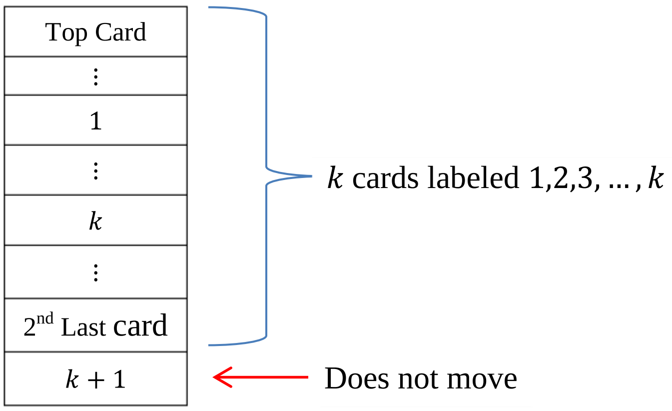
The card on the top is labeled 1 and the game is won.

Inductive Step:

Show: the game is eventually won with a deck of size k $\qquad (*)$
$\Rightarrow$ the game is eventually won with a deck of size $k+1$.

Case 1: The card labeled $k+1$ comes to the top.

After the card labeled $k+1$ comes to the top it gets placed on the bottom of the deck. The card labeled $k+1$ will stay on the bottom for the rest of the game. In this case the game is played with the remaining k cards from the top and is eventually won by $(*)$.

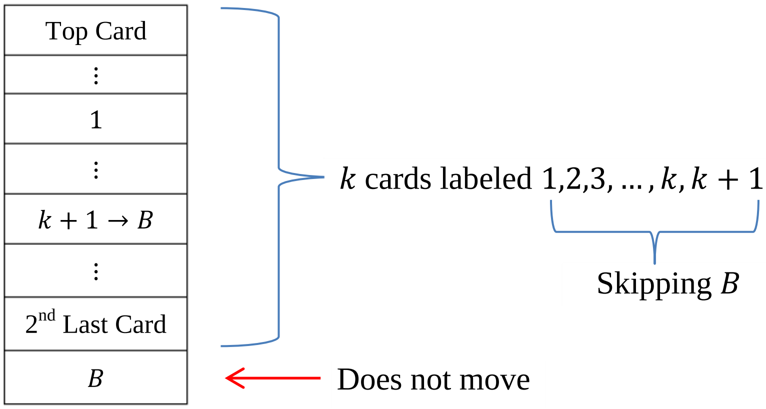


Case 2: The card labeled $k+1$ does not come to the top.

a) Suppose that the card labeled $B \neq 1$ is on the bottom of the deck. Since the card labeled $k+1$ does not come to the top throughout the game no card will be placed on the bottom of the deck throughout the game. This means the card labeled B will stay on the bottom for the rest of the game. In this case the game is played with the remaining k cards from the top:

$$\underbrace{1,2,3,\dots,k,k+1}_{\text{Skipping } B}$$

Since the card labeled $k+1$ does not come to the top it does not change the way the game is played. Therefore we can let the card labeled $k+1$ play the role of the card labeled B and play the game with a deck of k cards numbered from 1 to k. Now, the game is won by $(*)$.



b) Suppose that the card labeled $B=1$ is on the bottom of the deck. Follow the argument from part a). In doing so, we would let the card labeled $k+1$ play the role of the card labeled 1 and in turn the card labeled $k+1$ comes to the top. But, our assumption for this case is that the card labeled $k+1$ does not come to the top. Therefore Case 2 b) results in a contradiction and does not occur.

Another Solution. This game has 3 types of players:

- Goal Scorers are cards that come to the top of the deck.
- Defenders are cards that do not come to the top of the deck. Defenders move about the deck only taking up space; as a result they can change their name.
- Bench Warmers are cards that are stuck on the bottom of the deck. Bench Warmers do not affect the outcome so they can be removed from the game while they sit below the cards still in the game. Note: Bench Warmers are also Defenders except in the special case of a deck of size 1. In this special case card one is both a Goal Scorer and a Bench Warmer.

Step 1

Show: as the game is played every deck will produce a Bench Warmer unless the game is won beforehand. After a Bench Warmer is produced this game will be played with 1 less card.

There are 2 cases:

- Case 1. The largest card is a Goal Scorer. In this case the largest card will come to the top and become a Bench Warmer. Then this game can be played with 1 less card.
- Case 2. The largest card is a Defender. In this case there is a card stuck on the bottom of the deck; this card is a Bench Warmer. Now, the largest card and this Bench Warmer swap names and then our game is played with 1 less card.

Step 2

Show: card number one is a Goal Scorer.

Assume that card number one is a Defender. Then the game will continue being played forever; but as the game is played every deck will produce a Bench Warmer. After a Bench Warmer is produced this game will be played with 1 less card. This process (from step one) will continue on until all the cards are Bench Warmers. When all the cards are Bench Warmers the card on the top is both a Goal Scorer and a Bench Warmer. Card one is the only card capable of such a feat. This contradicts card one being a Defender.