

# MATH 222

## Assignment#4

Due: 5pm on the date stated in the course outline.  
Hand in to the assignment box on the 3<sup>rd</sup> floor of CAB.

1. Let  $a_n$  denote the number of interior regions of a convex polygon with  $n$  sides, divided by all its diagonals, if no three diagonals pass through a common point inside the polygon. Use the theorem below to show that:

$$a_n = \binom{n-1}{2} + \binom{n}{4}$$

for  $n \geq 4$ .

**Theorem:** Consider a convex region of the plane which is crossed by  $l$  lines with  $p$  interior points of intersection. No three lines pass through a common point inside the region. The number of disjoint regions created is:

$$1 + l + p$$

2. Use mathematical induction to prove that the sum of the first  $n$  square numbers is equal to  $\frac{n(n+1)(2n+1)}{6}$ . That is, prove:

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Conjecture a formula for the sum of the first  $n$  Fibonacci numbers with odd indices and prove your formula works by using mathematical induction. That is, find and prove a formula for

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1}$$

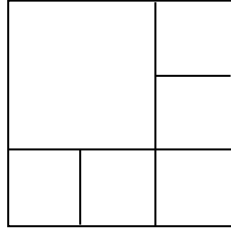
Note:

$$\begin{aligned} F_1 &= 1, \\ F_2 &= 1, \\ F_n &= F_{n-1} + F_{n-2}. \end{aligned}$$

4. An equal number of open and closed gas stations are distributed an equal distance apart around a ring road of a city. Say  $n$  stations are open and  $n$  stations are closed. Now suppose it takes  $n$  gallons of gas to travel around the city. If we start with an empty tank and can only add 1 gallon at a time at any open station, show that it possible to find a starting position to travel around the city clockwise for all  $n \geq 1$ ?

5. Prove that a square can be dissected into  $n$  smaller squares for any positive integer  $n \geq 6$ .  
(The smaller squares do not all have to be the same size)

For example a square can be dissected into 6 smaller squares:



6. Let  $a_n$  be the number of words of length  $n$  with digits from the set  $\{0,1,2\}$  with no consecutive 0's.

a) Show that:

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 8 \\ a_n &= 2a_{n-1} + 2a_{n-2} \end{aligned}$$

is a recurrence relation for  $a_n$ .

b) Use strong induction to show that:

$$a_n = \frac{(1 + \sqrt{3})^{n+2} - (1 - \sqrt{3})^{n+2}}{4\sqrt{3}}$$

is a solution to the recurrence relation from part a).

7. Prove Cassini's Identity:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

For  $n \geq 1$ .

Note:

$$\begin{aligned} F_1 &= 1, \\ F_2 &= 1, \\ F_n &= F_{n-1} + F_{n-2}. \end{aligned}$$

8. The objective of this question is to prove the following theorem.

For all natural numbers  $n \neq 5$  :

“a deficient  $n \times n$  board can be tiled with right trominoes”  $\iff n \not\equiv 0 \pmod{3}$ .

Recall: when  $n \equiv 1 \pmod{3}$  every deficient  $n \times n$  board can be tiled with trominoes.  
Finish proving the theorem by completing parts a) and b).

- a) Show that if  $n \equiv 0 \pmod{3}$  then a deficient  $n \times n$  board cannot be tiled with trominoes.
- b) Show that if  $n \equiv 2 \pmod{3}$  and  $n \neq 5$  then a deficient  $n \times n$  board can be tiled with trominoes.

9. For all  $n \geq 0$  show that any deficient  $14 \cdot 2^n \times 35 \cdot 2^n$  board can be tiled with right trominoes.

10. The principle of double induction is stated as follows.

If  $P(m, n)$  is a statement about the integers  $m \geq a$  and  $n \geq b$  such that

1.  $P(a, b)$  is true,
2. For all  $m \geq a$ , if  $P(m, b)$  is true, then  $P(m + 1, b)$  is true,
3. For all  $n \geq b$ , if  $P(m, n)$  is true for all  $m \geq a$ , then  $P(m, n + 1)$  is true for all  $m \geq a$ ,

then  $P(m, n)$  is true for all  $m \geq a$  and  $n \geq b$ .

Use the principle of double induction to show that for any integers  $m \geq 1$  and  $n \geq 1$  we have:

$$(m + 1)^n > mn$$

Bonus.



A game is played with a deck of  $k$  cards numbered from 1 to  $k$ . They are shuffled thoroughly and the top card is turned over. If it is number 1, the game is won. If it is number  $i$  where  $2 \leq i \leq k$ , then it is inserted into the deck so that it is the  $i$ -th card from the top. Then the new top card is turned over and the same process is applied. Can this game be won eventually, regardless of how the cards are stacked?