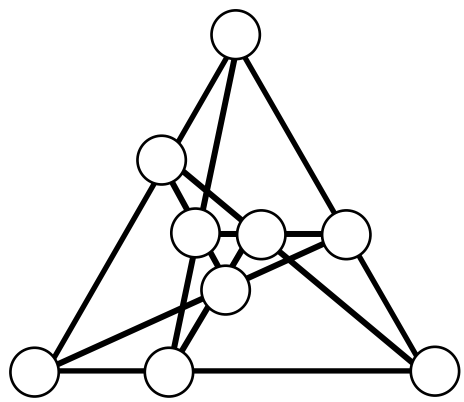
MATH 222

Assignment#2 Solutions

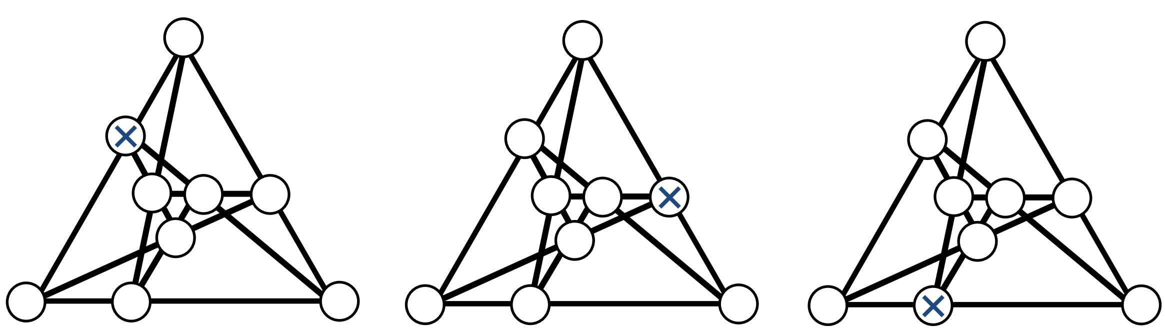
1. The following game is called Tri-Tic-Tack-Toe. How to play: Player X and Player O alternately put their symbol in an empty circle on their turn. Whoever makes three in a row on any straight line wins the game. Each player can only take four turns.



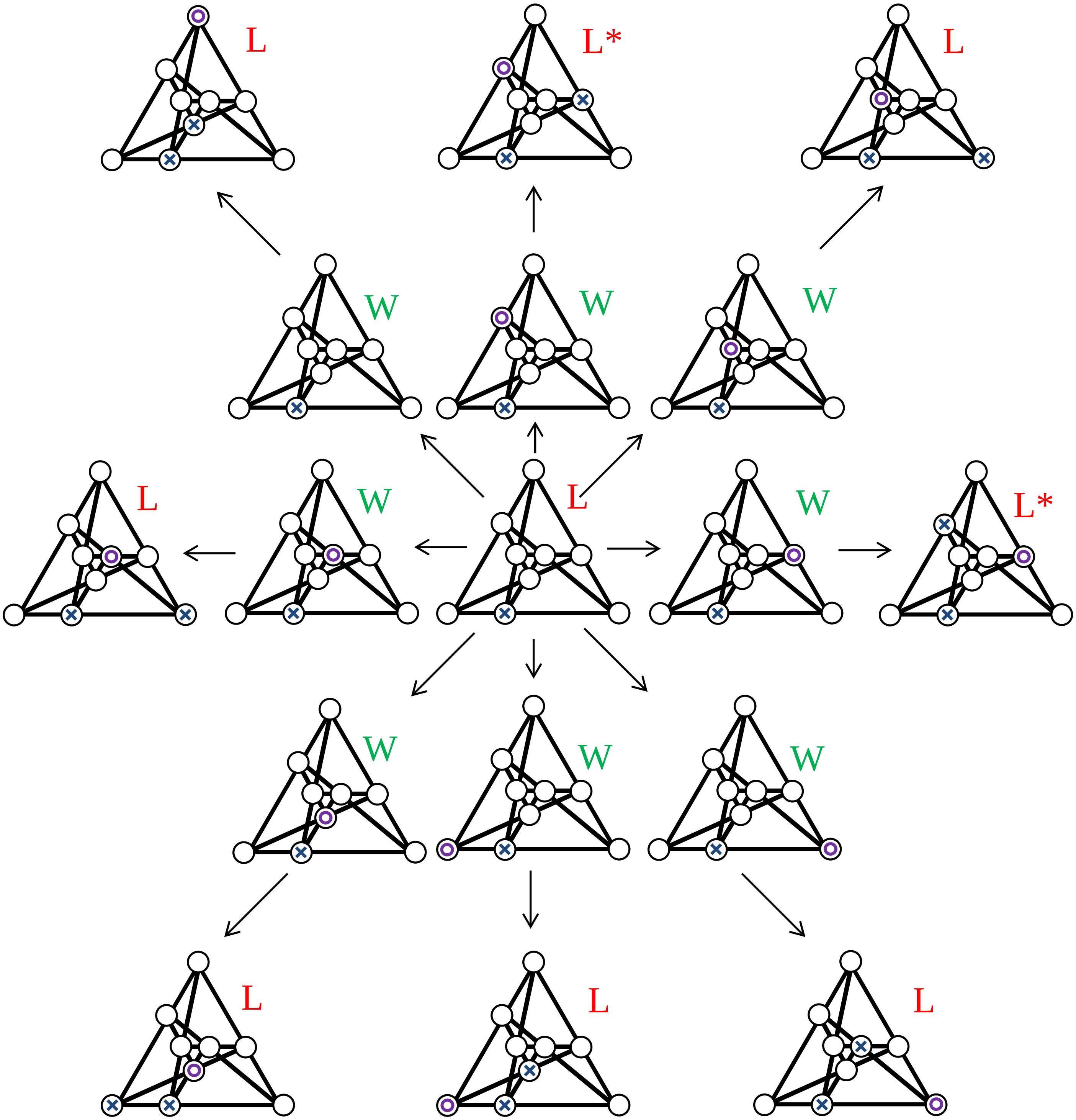
State whether Tri-Tic-Tack-Toe ends in a theoretical draw or if there is a winning strategy. If there is a winning strategy, which player can always win: the first player or the second player? If the first player can always win, state the first move that he or she needs to make. If the second player can always win, state the first move he or she must make in response the 9 ways the game may begin.

Solution:

In Tri-Tic-Tack-Toe there is a winning strategy for the 1st player (player X). There are 3 opening moves which follow the winning strategy:



For full marks you do not have to prove this. A proof using the 120° rotational symmetry of this result is given on the next page.



The first two moves of the winning strategy for the 1st player (player X) are outlined in the partial state diagram. In each "outer" losing position (except for the two marked with a *) player O is forced to block three X's in a row. Then player X plays so that there are two different ways to complete 3 X's in a row on the following turn; therefore, player X will win.

In the L* positions, player O can play anywhere; each play will force player X to block three O's in a row. In every case after blocking three O's in a row, player X has two different ways to get to get 3 X's in a row on the following turn; therefore, player X will win.

2. Let's play Nim. This version of the game will have 5 rows. The rows will have the following amount of counters:

Row 1: 30
Row 2: 31
Row 3: 42
Row 4: 76
Row 5: 104

a) Explain why the 1st player can always win this game.
b) Indicate all of the opening moves that can guarantee victory for the 1st player.

Solution:

a) Start by calculating the nim-sum of the game:

$$30 \oplus 31 \oplus 42 \oplus 76 \oplus 104 =$$
$$(\cancel{16} + \cancel{8} + \cancel{4} + \cancel{2}) \oplus (\cancel{16} + \cancel{8} + \cancel{4} + \cancel{2} + 1) \oplus (\cancel{32} + \cancel{8} + 2) \oplus (\cancel{64} + \cancel{8} + 4) \oplus (\cancel{64} + \cancel{32} + 8)$$
$$= 1 + 2 + 4 + 8$$
$$= 15$$
$$\neq 0$$

Since the nim-sum is non-zero the 1st player can always win.

b) The 1st player's objective is to set the nim-sum of the game to zero. Start with the nim-sum of this game and write it as a sum of distinct powers of 2:

$$1 + 2 + 4 + 8$$

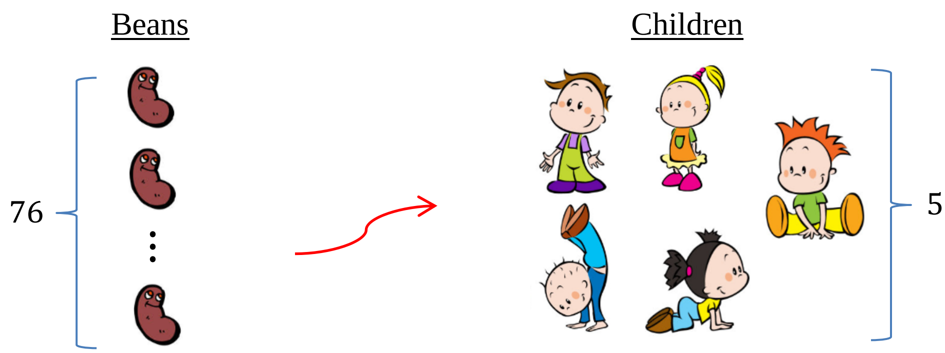
The largest power of 2 appearing in this sum is 8. Search for a row that has 8 in its distinct sum of powers of 2. Since every row meets this criterion, there is an opening move in every row that can guarantee victory for the 1st player. The moves for each row are given in the chart below:

Row	Counters	Number Of Counters After The Opening Move
1	30	$15 \oplus 30 = (1 + \cancel{2} + \cancel{4} + \cancel{8}) \oplus (16 + \cancel{8} + \cancel{4} + \cancel{2}) = 17$
2	31	$15 \oplus 31 = (\cancel{1} + \cancel{2} + \cancel{4} + \cancel{8}) \oplus (16 + \cancel{8} + \cancel{4} + \cancel{2} + \cancel{1}) = 16$
3	42	$15 \oplus 42 = (1 + \cancel{2} + 4 + \cancel{8}) \oplus (32 + \cancel{8} + \cancel{2}) = 37$
4	76	$15 \oplus 76 = (1 + 2 + \cancel{4} + \cancel{8}) \oplus (64 + \cancel{8} + \cancel{4}) = 67$
5	104	$15 \oplus 104 = (1 + 2 + 4 + \cancel{8}) \oplus (64 + 32 + \cancel{8}) = 103$

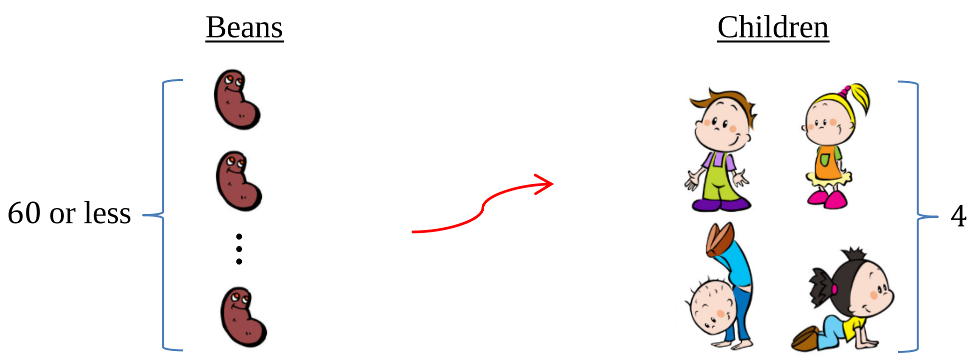
Let's prove that there are no other opening moves that follow a winning strategy. We will consider row 1 since the argument is the same for every other row. Suppose there is a different move on row 1 which follows a winning strategy; it too must set the nim-sum to zero. Let's say the game is in state A after the alternative opening move on row 1 is made. Let's say the game is in state B after row 1 is changed from 30 to 17 counters. Consider a move from state A to state B or vice-versa (whichever is a valid move). Before this move, the nim-sum is zero and after this move, the nim-sum is zero. Such a move is impossible since in the game of Nim, every move changes the nim-sum. This shows that another successful opening move results in a contradiction and therefore cannot exist.

3. There were 5 children in the Emergency Room and between them they had stuck a total of 76 beans up their noses. Show that there must be three children with a combined total of 46 or more beans up their noses.

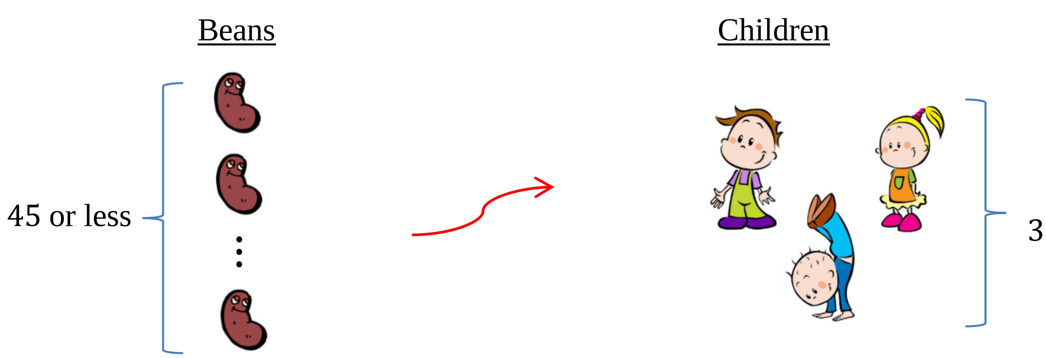
Solution:



Therefore one child has $\left\lceil \frac{76}{5} \right\rceil = 16$ or more beans in their nose. Equivalently; the other four have $76 - 16 = 60$ or less beans in their noses:



Therefore, three children have $60 - \left\lceil \frac{60}{4} \right\rceil = 45$ or less beans in their noses:



Therefore, two children have $45 - \left\lceil \frac{45}{3} \right\rceil = 30$ or less beans in their noses. Equivalently; the other three have $76 - 30 = 46$ or more beans in their noses.

Another Solution: Assume that every combination of three children has a combined total of 45 or less beans up their noses. Let the number of beans in the five children's noses be: a, b, c, d, e respectively. Now

$$\begin{array}{lcccccl}
 & a & +b & +c & & & \leq 45 \\
 & a & +b & & +d & & \leq 45 \\
 & a & +b & & & +e & \leq 45 \\
 & a & & +c & +d & & \leq 45 \\
 & a & & +c & & +e & \leq 45 \\
 & a & & & +d & +e & \leq 45 \\
 & & b & +c & +d & & \leq 45 \\
 & & b & +c & & +e & \leq 45 \\
 & & b & & +d & +e & \leq 45 \\
+ & & & c & +d & +e & \leq 45 \\
\hline
\end{array}$$

$$456 = 6(76) = 6(a + b + c + d + e) \leq 45(10) = 450$$

This is a contradiction. Therefore there must be one combination of three children having a combined total of 46 or more beans up their noses.

4. Prove that there exists two powers of two which differ by a multiple of 2020.

Solution:

In the lecture notes it was shown that:

In a collection of $n + 1$ distinct integers, there are distinct integers x and y such that $x - y$ is a multiple of n.

Let $n = 2020$ and consider the following collection of $n + 1 = 2021$ distinct integers:

$$2^0, 2^1, \dots, 2^{2021}$$

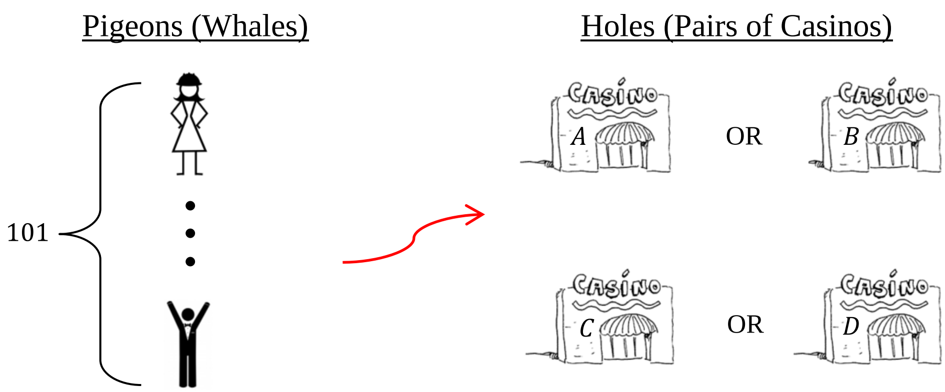
Therefore, there are two powers of two: 2^i and 2^j such that $2^i - 2^j$ is a multiple of 2020.

5. In Las Vegas on a particular weekend night there are 101 whales (a nickname given to high stakes gamblers) who randomly pick four different casinos to gamble at.

a) Prove that there is a pair of casinos that will host a combined total of at least 51 whales.

b) Out of the four casinos we can make a total of 6 pairs of casinos. How many pairs out of the 6 pairs will host a total of at least 51 whales? Your answer **must** include the quantifier "at least", "exactly" or "at most".

Solution:
a) Let the four casinos be A, B, C, D. Using the pigeon hole principle, we let the 101 whales be pigeons that fly into two different holes. The first hole will represent a whale going to casino A or B while the second hole will represent a whale going to casino C or D. Consider the picture:



This means that there will be a total of at least 51 whales at the pair A, B or there will be a total of at least 51 whales at the pair C, D. Therefore there is a pair of casinos that will host a total of at least 51 whales.

b) There is a total of three different ways we could pick the holes in part a)

- Choice 1: The first hole will represent a whale going to casino A or B while the second hole will represent a whale going to casino C or D.
- Choice 2: The first hole will represent a whale going to casino A or C while the second hole will represent a whale going to casino B or D.
- Choice 3: The first hole will represent a whale going to casino A or D while the second hole will represent a whale going to casino B or C.

In each of the above choices **exactly one** pair will host a total of at least 51 whales. Therefore out of the six pairs **exactly three** will host a total of at least 51 whales.

6. A counselor has $3n + 2$ campers with her at a junction in a hiking trail. She knows their camp is twenty minutes down one of four possible paths. It will be dark in one hour and the group must find their camp before dark. n of the $3n + 2$ campers sometimes lie, and unfortunately the counselor doesn't know which n they are.

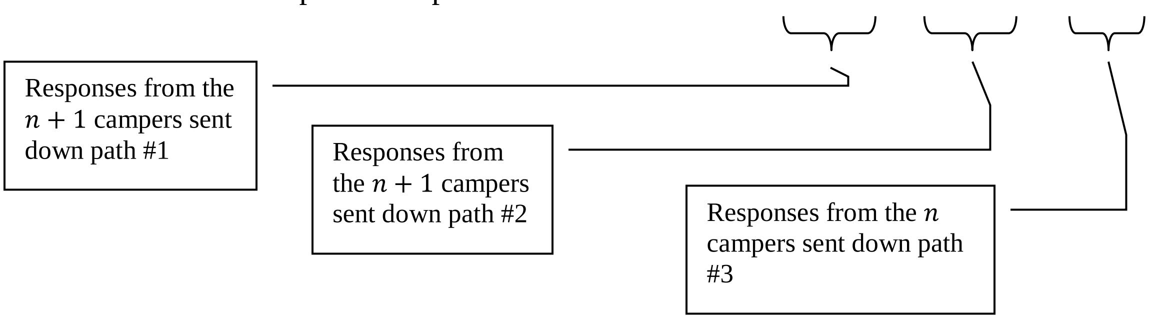
   a) Is there a way for the counselor to accurately deduce the location of the camp?

   b) Prove this problem cannot be solved with only $3n + 1$ campers.

Solution:

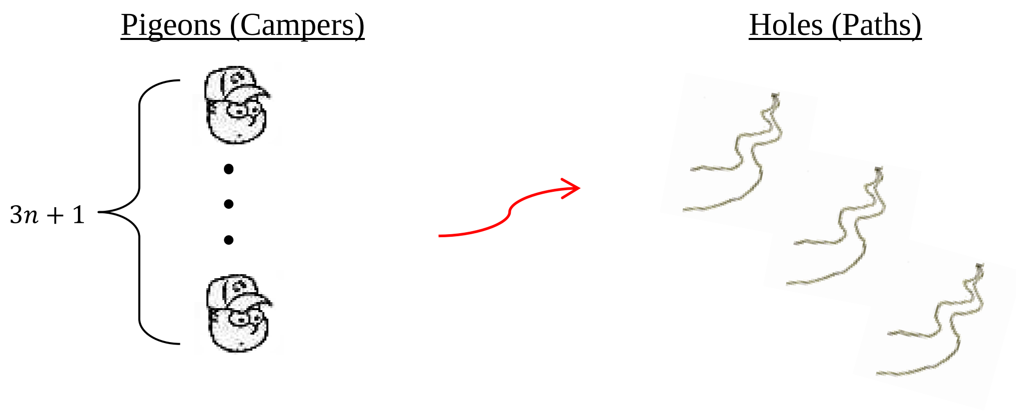
a)
Make a code containing three codewords. Each codeword will represent the camp being down a particular path. Each bit in a codeword will represent the answer given by a camper.
In this code a 1 will represent yes and a 0 will represent no.

Codeword #1: The camp is down path one and no one lies: $11\cdots 11 \quad 00\cdots 00 \quad 00\cdots 0$
Codeword #2: The camp is down path two and no one lies: $00\cdots 00 \quad 11\cdots 11 \quad 00\cdots 0$
Codeword #3: The camp is down path three and no one lies: $00\cdots 00 \quad 00\cdots 00 \quad 11\cdots 1$



Finally, since each pair of codewords in the code has a minimum hamming distance of $2n + 1$ the code can correct up to n corrupted bits (liars). Therefore counselor can accurately deduce the location of the camp.

b) We start by showing that there is a pair of paths that has a combined total of at most $2n$ campers.



By the Pigeonhole Principle, one path has at least $n + 1$ campers on it. It follows that the other paths have a combined total of at most $2n$ campers.

When these two paths are represented as codewords as it was done in part a) the Hamming distance between these two paths can be at most $2n$. But if any two codewords have a Hamming distance of less than $2n + 1$ then we can no longer correct n errors.

7. A different subset of 32 natural numbers less than 64 is printed on each of the 6 cards below according to the following rules:

   Card 1: The 1st digit in the binary expansion of each number on this card is a one.
   Card 2: The 2nd digit in the binary expansion of each number on this card is a one.
   Card 3: The 3rd digit in the binary expansion of each number on this card is a one.
   Card 4: The 4th digit in the binary expansion of each number on this card is a one.
   Card 5: The 5th digit in the binary expansion of each number on this card is a one.
   Card 6: The 6th digit in the binary expansion of each number on this card is a one.

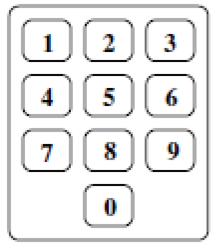
   Note: the 1st (rightmost) digit in the binary expansion appears in the rightmost position while the 6th digit in the binary expansion appears in the leftmost position.

   I am looking at a number that appears on cards 1, 3, and 6 but not on cards 2, 4, and 5. What number am I looking at?

Solution. The number you are looking at is 100101 in binary: which is 37 in decimal.

8. The telephone numbers in a town run from 00000 to 99999. There are a number of errors that commonly occur:

- <u>The Diagonal Error</u>: a common error in dialling on a standard keypad is to punch in a digit diagonally adjacent to the intended one. So on a standard dialling keypad, 4 could erroneously be entered as 2 or 8 (but not as 1, 5, or 7).



- <u>The Switch Bug</u>: After a correct number has been called, in transmission, one pair of adjacent digits gets swapped. For example the number $abcde$ could be called but whoever is at $abc\boldsymbol{ed}$ receives the phone call.

- <u>The Smudged Digit</u>: A single digit gets corrupted in transmission and it is known which digit got corrupted. For example: 38754 is dialed but in transmission, it becomes 387□4.

- <u>The Digit Error:</u> A single digit gets changed into another random digit during transmission and it is unknown which digit was changed. For example 38754 is dialed but 38757 receives the call.

It has been decided that a sixth digit will be added to each phone number. Given a phone number $abcde$, there are three different proposed schemes:

Code 1: $a+b+c+d+e+X \equiv 0 \pmod{10}$

Code 2: $2a+b+2c+d+2e+X \equiv 0 \pmod{10}$

Code 3: $6a+5b+4c+3d+2e+X \equiv 0 \pmod{10}$

Fill out the following chart with a Yes or a No (you DO NOT have to show your work).

	Can detect a Diagonal Error?	Can detect a Switch Bug?	Can correct a Smudged Digit?	Can detect a Digit Error?
Code 1	Yes	No	Yes	Yes
Code 2	Yes	Yes	No	No
Code 3	No	Yes	No	No

Solution.

Code 1:

Given a phone number $abcde$ a sixth digit X is added so:

$$a + b + c + d + e + X \equiv 0 \pmod{10}$$

- Can detect a Diagonal Error? Yes, by the same reasoning Code 1 can detect a Digit Error (below).

- Can detect a Switch Bug? No, for example the switch bug could change 100009 into 010009 both of which are valid codewords.

- Can correct a Smudged Digit? Yes, for example if $a \leftrightarrow y$ then we can solve

$$y + b + c + d + e + X \equiv 0 \pmod{10}$$

To get

$$y \equiv -b - c - d - e - X \pmod{10}$$

The smudge y can be corrected to $a = -b - c - d - e - X$

- Can detect a Digit Error? Yes, for example if $a \leftrightarrow a'$

Case 1: $a' + b + c + d + e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the digit error is detected.

Case 2: $a' + b + c + d + e + X \equiv 0 \pmod{10}$

Then

$$\begin{array}{rl} & a' + b + c + d + e + X \equiv 0 \pmod{10} \\ - & a + b + c + d + e + X \equiv 0 \pmod{10} \\ \hline & a' - a \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow a' \equiv a \pmod{10}$

$\Rightarrow a' = a$ since $a, a' \in \{0,1,2,\ldots,9\}$

$\therefore$ in this case there was no error.

Code 2:

Given a phone number $abcde$ a sixth digit X is added so:

$$2a + b + 2c + d + 2e + X \equiv 0 \pmod{10}$$

- Can detect a Diagonal Error? Yes. There are two situations:

  1. When a digit with a coefficient of 2 gets corrupted for example $a \leftrightarrow a'$.
  2. When a digit with a coefficient of 1 gets corrupted for example $b \leftrightarrow b'$.

  The same reasoning code 1 can detect a Digit Error (above) shows that code 2 can detect when $b \leftrightarrow b'$. We are left to discuss when $a \leftrightarrow a'$.

Case 1: $2a' + b + 2c + d + 2e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the Diagonal Error is detected.

Case 2: $2a' + b + 2c + d + 2e + X \equiv 0 \pmod{10}$

Then

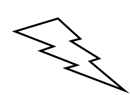
$$\begin{array}{rl} & 2a' + b + 2c + d + 2e + X \equiv 0 \pmod{10} \\ - & 2a + b + 2c + d + 2e + X \equiv 0 \pmod{10} \\ \hline & 2a' - 2a \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow 2(a' - a) \equiv 0 \pmod{10}$

$\Rightarrow (a' - a) \equiv 0 \text{ or } 5 \pmod{10}$

Now, assume that a Diagonal Error occurred and consider all the possibilities:

$$a' - a = \pm 2, \pm 4, \pm 7, \pm 9$$

$\Rightarrow (a' - a) \equiv 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 6 \text{ or } 7 \text{ or } 8 \text{ or } 9 \pmod{10}$ 

$\therefore$ in this case there was no error.

- Can detect a Switch Bug? Yes. For example if $a \leftrightarrow b$

Case 1: $2b + a + 2c + d + 2e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the digit error is detected.

Case 2: $2b + a + 2c + d + 2e + X \equiv 0 \pmod{10}$

Then

$$\begin{array}{rl} & 2b + a + 2c + d + 2e + X \equiv 0 \pmod{10} \\ - & 2a + b + 2c + d + 2e + X \equiv 0 \pmod{10} \\ \hline & b - a \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow b \equiv a \pmod{10}$

$\Rightarrow b = a$ since $a, a' \in \{0,1,2,\dots,9\}$

$\therefore$ in this case there was no error.

- Can correct a Smudged Digit? No, for example the smudged phone number □00000 could have been correctly dialed as 000000 or 500000; both of which are valid codewords.

- Can detect a Digit Error? No, the valid codewords 000000 or 500000 have a Hamming distance of 1. Therefore, the minimum Hamming distance of Code 2 is one, which means this code can't detect a Digit Error.

Code 3:

Given a phone number $abcde$ a sixth digit X is added so:

$$6a + 5b + 4c + 3d + 2e + X \equiv 0 \pmod{10}$$

- Can detect a Diagonal Error? No, for example a Diagonal Error could change 020000 into 040000 both of which are valid codewords.

- Can detect a Switch Bug? Yes. For example if $a \leftrightarrow b$

Case 1: $6b + 5a + 4c + 3d + 2e + X \not\equiv 0 \pmod{10}$

$\therefore$ in this case the digit error is detected.

Case 2: $6b + 5a + 4c + 3d + 2e + X \equiv 0 \pmod{10}$

Then

$$\begin{array}{rl} & 6b + 5a + 4c + 3d + 2e + X \equiv 0 \pmod{10} \\ - & 6a + 5b + 4c + 3d + 2e + X \equiv 0 \pmod{10} \\ \hline & b - a \equiv 0 \pmod{10} \end{array}$$

$\Rightarrow b \equiv a \pmod{10}$

$\Rightarrow b = a$ since $a, a' \in \{0,1,2,\dots,9\}$

$\therefore$ in this case there was no error.

- Can correct a Smudged Digit? No, for example the smudged phone number $0000\square0$ could have been correctly dialed as 000000 or 000050; both of which are valid codewords.

- Can detect a Digit Error? No, the valid codewords 000000 or 000050 have a Hamming distance of 1. Therefore, the minimum Hamming distance of Code 3 is one, which means this code can't detect a Digit Error.

9. The following message is received using the 15-digit Hamming Code. Correct any single error that may have occurred during transmission.

a	*a*	*a*	*a*		*a*	*a*	*a*				*a*			
b	*b*	*b*		*b*	*b*			*b*	*b*			*b*		
c	*c*		*c*	*c*		*c*		*c*		*c*			*c*	
d		*d*	*d*	*d*			*d*		*d*	*d*				*d*
0	1	0	1	1	0	1	0	0	0	0	0	1	1	0

Solution: There is an error in the column containing *a*, *b*, and *c*. The codeword that was sent was:

0 0 0 1 1 0 1 0 0 0 0 0 1 1 0

10. I am thinking of a number between 0 and 15. You ask me to write the digit in binary form (using 4 digits) and ask the following seven questions.

Is the 1st (rightmost) digit in the binary expansion of your number a one?

Is the 2nd digit in the binary expansion of your number a one?

Is the 3rd digit in the binary expansion of your number a one?

Is the 4th digit in the binary expansion of your number a one?

Is there an odd number of ones in the 2nd, 3rd, and 4th positions?

Is there an odd number of ones in the 1st, 3rd, and 4th positions?

Is there an odd number of ones in the 1st, 2nd, and 4th positions?

My answers to your questions in the order you asked them are:

No, Yes, Yes, No, Yes, No, No.

However, I may have lied once. What is my number?

Solution: This is equivalent to receiving the following word with the Hamming code:

a	*a*	*a*		*a*		
b	*b*		*b*		*b*	
c		*c*	*c*			*c*
0	1	1	0	1	0	0

There is an inconsistency in the column containing *a*, *b*, and *c*. Thus, the codeword (without lying) should have been:

1 1 1 0 1 0 0

The number I am thinking of is 1110 in binary: which is 14 in decimal.

Bonus question.

A different subset of the natural numbers less than 64 is printed on each of the 6 cards below according to the following rules:

Card 1: The 1^{st} (rightmost) digit in the binary expansion of each number on this card is a one.
Card 2: The 2^{nd} digit in the binary expansion of each number on this card is a one.
Card 3: The 3^{rd} digit in the binary expansion of each number on this card is a one.
Card 4: The 4^{th} digit in the binary expansion of each number on this card is a one.
Card 5: The 5^{th} digit in the binary expansion of each number on this card is a one.
Card 6: The 6^{th} digit in the binary expansion of each number on this card is a one.

Dr. Ecco found a twin prime number that appears on cards 1, 3, and 6 but not on cards 2, 4, and 5. Find Dr. Ecco's number.

Solution:

Dr. Ecco's number appears on cards 1, 3, and 6 therefore, Dr. Ecco's number in binary has a 1 in the 1^{st}, 3^{rd}, and 6^{th} positions. It is unknown why Dr. Ecco's number is not on cards 2, 4, and 5. For example, his number in binary could have a 0 in the 2^{nd} position and contradict the rule on card 2 or his number in binary might have a 1 in the 2^{nd} position and be left out of the subset on card 2.

There are 8 choices for Dr. Ecco's number:

100101 – 37
100111 – 39
101101 – 45
101111 – 47
110101 – 53
110111 – 55
111101 – 61
111111 – 63

of which, only 61 is a twin prime. Therefore, Dr. Ecco's number is 61.