MATH 222

Assignment#1 – Solutions

1. Four people are being pursued by a menacing beast. It is nighttime, and they need to cross a bridge to reach safety. It is pitch black, and only two can cross at once. They need to carry a lamp to light their way. The first person can cross the bridge in no less than 10 minutes, the second in 5 minutes, the third in 2 minutes, and the fourth in 1 minute. If two cross together, the couple is only as fast as the slowest person. (That is, a fast person can't carry a slower person to save time, for example. If the 10-minute person and the 1-minute person cross the bridge together, it will take them 10 minutes.) The person or couple crossing the bridge needs the lamp for the entire crossing and the lamp must be carried back and forth across the bridge (no throwing, etc.) If they don't all get completely across in less than 18 ½ minutes, whoever is on the bridge or left behind will be eaten by the beast. Is it possible for all of them to get across?

Solution: One solution is as follows: Label the people 1, 2, 5, and 10, in accordance with the time it takes each to cross the bridge, then

1 and 2 cross	(2 minutes)
2 returns with the lantern	(2 minutes)
5 and 10 cross	(10 minutes)
1 returns with the lantern	(1 minute)
1 and 2 cross	(2 minutes)

2. Find the multiplicative inverse of 13 in mod 30.

Solution:

$1 \equiv 31 \equiv 61 \equiv 91 \equiv 7 \cdot 13 \pmod{30}$

$\Rightarrow$ 7 is the inverse of 13 in mod 30.

3. There is only one way to open the safe below.

4D	4D	1L	3L	Open
2R	1D	1U	2L	4L
4R	1L	2D	1U	2L
4R	2R	2L	1D	2U
4R	1U	1U	4U	4U

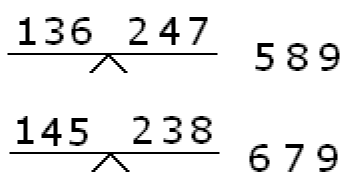
You must press each button exactly once in the correct order in order to reach Open. Each button is marked with a direction: U is up, L is left, R is right, and D is down. The number of spaces to move is also marked on each button. Which button is the first one you must press?

Solution: The first button you must push is 1U in row 3, column 4.

4. There are 9 coins, all identical except that one is counterfeit and is a heavier than the others. Show how to find the counterfeit in two weighings using a pan balance.

Solution: -Let R mean the scale tips to the right.
-Let L mean the scale tips to the left.
-Let B mean the scale is balanced.

Weigh the coins in the following pattern:



Now:

LL ⇒ coin 1

RR ⇒ coin 2

LR ⇒ coin 3

RL ⇒ coin 4

BL ⇒ coin 5

LB ⇒ coin 6

RB ⇒ coin 7

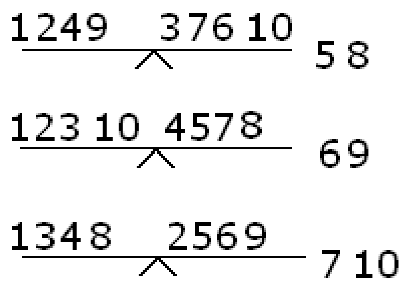
BR ⇒ coin 8

BB ⇒ coin 9

5. There are 10 coins, all identical except that one is counterfeit and is a different weight than the others. It is not known whether the counterfeit is heavier or lighter. Show how to find the counterfeit in three weighings using a pan balance.

Solution: -Let R represent the scale tipping to the right.
-Let L represent the scale tipping to the left.
-Let B represent when the scale is balanced.

Weigh the coins in the following pattern:



Now:
LLL or RRR ⇒ coin 1
LLR or RRL ⇒ coin 2
RLL or LRR ⇒ coin 3
LRL or RLR ⇒ coin 4
BRR or BLL ⇒ coin 5
RBR or LBL ⇒ coin 6
RRB or LLB ⇒ coin 7
BRL or BLR ⇒ coin 8
LBR or RBL ⇒ coin 9
RLB or LRB ⇒ coin 10

6. Determine if each of the following pairs of integers are congruent modulo 11.

a) 2 , 218
b) 0 , 242
c) 4 , 420
d) -7 , 7
e) -31 , 68
f) -1 , 120

Solution:

Since $218 = 19 \cdot 11 + 9$

a) $218 \equiv 9 \not\equiv 2 \bmod 11$ No

Since $242 = 22 \cdot 11$

b) $242 \equiv 0 \bmod 11$ Yes

Since $420 = 38 \cdot 11 + 2$

c) $420 \equiv 2 \not\equiv 4 \bmod 11$ No

Since $-7 = -1 \cdot 11 + 4$

d) $-7 \equiv 4 \not\equiv 7 \bmod 11$ No

Since $-31 = -3 \cdot 11 + 2$ and $68 = 6 \cdot 11 + 2$

e) $-31 \equiv 2 \equiv 68 \bmod 11$ Yes

Since $-1 = -1 \cdot 11 + 10$ and $120 = 10 \cdot 11 + 10$

f) $-1 \equiv 10 \equiv 120 \bmod 11$ Yes

7. In 2020, Valentine's Day is on a Friday. What day will Valentine's Day be in the year 3000? (Don't forget that every year divisible by 4 is a leap year unless it is divisibly by 100. That is, 2100, 2200, 2300, etc are not leap years. An exception is that centuries are leap years when they are divisible by 400; so 2000 was a leap year and so will be 2400). Hint: There will be 238 occurrences of February 29th between February 14th 2020 and February 14th 3000.

Solution:

There will be a total of $365 \cdot 980 + 238$ days between February 14th 2020 and February 14th 3000. Letting Friday be 5 in mod 7 we reduce:

$$\begin{aligned} & 5 + 365 \cdot 980 + 238 \\ & \equiv 5 + 1 \cdot 0 + 0 \\ & \equiv 5 \pmod 7 \end{aligned}$$

Since, we get 5 in mod 7; Valentine's Day will be a Friday in the year 3000.

8. Show that none of the integers:

$$11, 111, 1111, 11111, \ldots$$

is the square of another integer.

Hint: Pick any number from the above list:

$$111 \cdots 11$$

Now note that $111 \cdots 11 \equiv 111 \cdots 100 + 11 \equiv 0 + 3 \equiv 3 \pmod 4$.

Solution:

Since every integer is congruent to either $0, 1, 2$ or 3 in mod 4 the square of every integer in mod 4 congruent to either:

$$\begin{aligned} 0^2 &\equiv 0 \pmod 4 \\ 1^2 &\equiv 1 \pmod 4 \\ 2^2 &\equiv 0 \pmod 4 \\ 3^2 &\equiv 1 \pmod 4 \end{aligned}$$

Therefore if we let n be any integer we have:

$$\begin{aligned} n^2 &\equiv 0 \text{ OR } 1 \\ &\not\equiv 3 \\ &\equiv 111 \cdots 11 \pmod 4 \end{aligned}$$

Therefore $111 \cdots 11 \not\equiv n^2 \pmod 4$ and we can conclude that $111 \cdots 11 \neq n^2$. Which means an integer of the form $111 \cdots 11$ is not a square of another integer.

9. Five darts are thrown at a square target measuring 14 inches on a side. Prove that two of them must be at a distance no more than 10 inches apart.

Solution: Divide the target into 4 squares, each measuring 7 inches on a side, and throw 5 darts at the target. By the pigeonhole principle, one of the smaller squares must contain at least 2 of the darts, and the distance between these two darts must be less than or equal to

$$\sqrt{7^2 + 7^2} = \sqrt{49 + 49} = \sqrt{98}$$

Since $\sqrt{98} < 10$ there are at least two darts which are at a distance no more than 10 inches apart.

10. There are n coins of random integer values coin lined up in a row on a table. Nora picks a coin from one of the ends of the row. Next, Abraham picks up a coin from one of the ends of the remaining row of coins. They alternate in this manner until the last coin is picked up. Nora wins the game if she has picked up at least the same amount of money as Abraham.

   a. Find all the positive integer values for n (the number of coins) where Nora can always win regardless of the values of each coin. Provide a reason why.

   b. Find all the positive integer values for n where Nora cannot always win regardless of the values of each coin. To show that Nora cannot always win provide a counter example with specific values for each of the n coins where Abraham can always win.

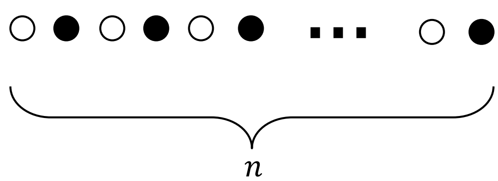
Solution:

a)

Case 1: $(n = 1)$ Nora can always win by picking up the single coin.

Case 2: (n is even)

Checker the coins white and black so that white coins are adjacent to only black coins (and vice versa):



Notice that if Nora takes a white coin first Abraham must take a black coin second. She can continue to take all the white coins forcing Abraham to take all the black coins. Similarly, Nora can take all the black coins.

Therefore when n is even, Nora can always win by taking the larger of the two groups:

- Group 1: the sum of the white coins
- Group 2: the sum of the black coins

b)

Case 3: ($n > 1$ is odd)

When $n > 1$ is odd, Nora cannot always win. Consider the coins being placed in the pattern:

$$13\ 13\ 13 \ldots 13\ 1$$

With this pattern Abraham can always take all the 3’s and win.

Bonus question

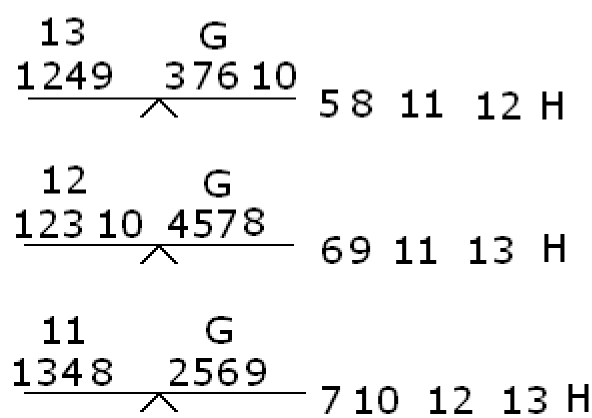
Suddenly a stern knock on Dr. Ecco's door, and in walked Michael Monetary. The man worked inside the government and was in charge of a coin making factory. He stated his problem: "one of my coin making machines is not working properly. It produces coins in batches of 15; exactly one out of every batch has an incorrect weight. The first coin from a batch is always perfect, the second if it is the one of incorrect weight is never too light, and otherwise the bad coin could end up being heavier or lighter than it's supposed to be. I would like my workers to quickly find and remove the bad coin from each batch using a regular pan balance in 3 weighings. Anymore weighings and I will surely lose my job over production losses. Finally, I would like to keep track of the bad coins being too heavy or to light; this statistic could help fix the machine."

After a few minutes Ecco handed Mr. Monetary a scrap of paper and said "here is your solution in three weighings, it's a good thing the first coin from a batch is always perfect."

How did Ecco solve Mr. Monetary's problem?

Solution: -Let R represent the scale tipping to the right.
-Let L represent the scale tipping to the left.
-Let B represent when the scale is balanced.
-Let G be the coin that is always perfect.
-Let H be the coin that cannot be too light.

Weigh the coins in the following pattern:



Now:

LLL (heavy) or RRR (light) ⇒ coin 1
LLR (heavy) or RRL (light) ⇒ coin 2
RLL (heavy) or LRR (light) ⇒ coin 3
LRL (heavy) or RLR (light) ⇒ coin 4
BRR (heavy) or BLL (light) ⇒ coin 5
RBR (heavy) or LBL (light) ⇒ coin 6
RRB (heavy) or LLB (light) ⇒ coin 7
BRL (heavy) or BLR (light) ⇒ coin 8
LBR (heavy) or RBL (light) ⇒ coin 9
RLB (heavy) or LRB (light) ⇒ coin 10
BBL (heavy) or BBR (light) ⇒ coin 11
BLB (heavy) or BRB (light) ⇒ coin 12
LBB (heavy) or RBB (light) ⇒ coin 13
BBB (heavy) ⇒ coin H