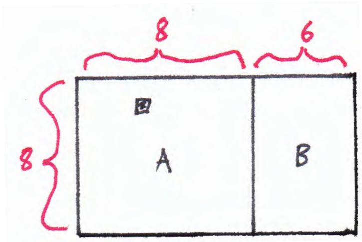
MATH 222

Final Practice Problems

1. For all $n \geq 0$, show that any deficient $(2^n \cdot 8) \times (2^n \cdot 14)$ board can be tiled with right trominoes.

<u>BC</u> $(n=0)$



· By symmetry the missing square is in section A.

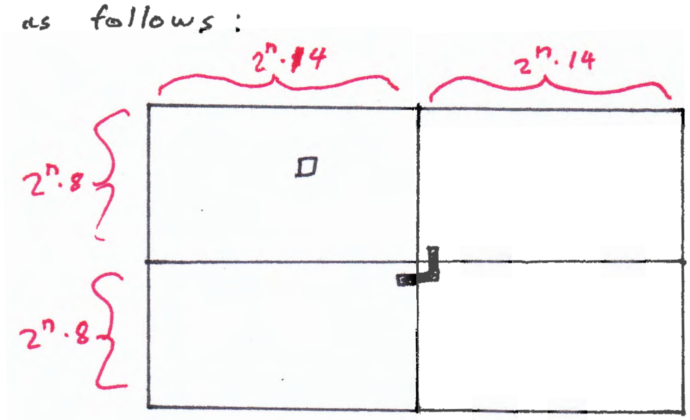
· Section A can be tiled by assignment 4.

· Section B " " " " L 16.

<u>IS</u> Show: any deficient $2^n \cdot 8 \times 2^n \cdot 14$ board can be tiled (*)

$\Longrightarrow$ " " $2^{n+1} \cdot 8 \times 2^{n+1} \cdot 14$ " " " ", for $n \geq 0$.

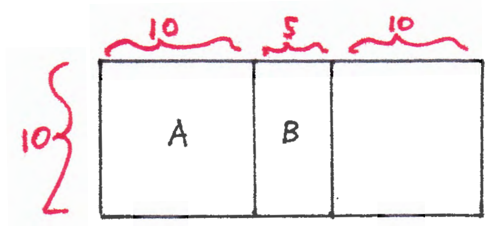
· By symmetry the missing square is in the top left. Secton the board as follows:



Now placing one tromino in the center as shown allows us to tile the rest of the board by (*).

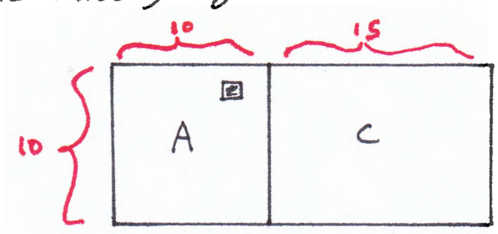
2. Show that any deficient 25×10 board can be tiled with right trominoes.

By symmetry the missing square is on the left side of the board:



in section A or B.

CASE 1 (the missing square is in section A)



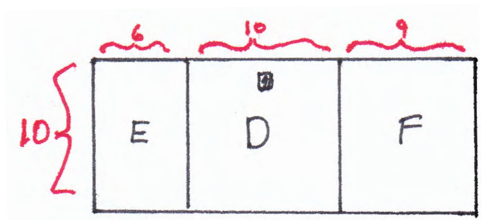
Section A can be tiled since $10 \equiv 1 \not\equiv 0 \mod 3$.

" C " " " " $15 \equiv 0 \mod 3$

$10 \equiv 0 \mod 2$.

CASE 2 (the missing square is in section B)

Place a 10×10 block around section B:



Section D can be tiled since $10 \equiv 1 \not\equiv 0 \mod 3$.

" E " " " " $6 \equiv 0 \mod 3$

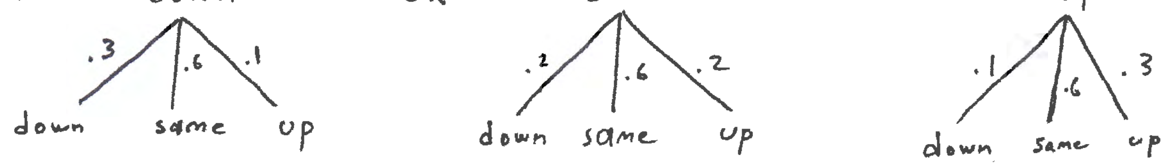
$10 \equiv 0 \mod 2$.

" F " " " " $9 \equiv 0 \mod 3$

$10 \equiv 0 \mod 2$.

3. If the price of gold does not change on a certain day, it will remain the same the next day with a probability of 0.6, go down with 0.2 and go up with 0.2. If it goes down on a certain day, it will continue to do so the next day with a probability of 0.3, stay put with 0.6 and go up with 0.1. If it goes up on a certain day, it will continue to do so the next day with a probability of 0.3, stay put with 0.6 and go down with 0.1. During a certain week, the price goes up on Monday. What is the probability that it will go down on Friday?

day n: down OR same OR up



let s_n = probability the price of gold will stay the same on day n,

u_n = ―――――――――――――― go up ――――,

d_n = ―――――――――――――― go down ――――.

Note: $s_n = (.6)d_{n-1} + (.6)s_{n-1} + (.6)u_{n-1}$

$= (.6)(d_{n-1} + s_{n-1} + u_{n-1})$

$= (.6)\cdot 1$

$= .6$

Now, $d_n = (.3)d_{n-1} + (.2)s_{n-1} + (.1)u_{n-1}$

$= (.3)d_{n-1} + (.2)(.6) + (.1)(1-(.6)-d_{n-1})$ (by note)

$= (.2)d_{n-1} + .12 + .04$

$= (.2)d_{n-1} + .16$

TUE: $d_1 = 0.1$

WED: $d_2 = (.2)(.1) + (.16) = 0.18$

THUR: $d_3 = (.2)(.18) + (.16) = 0.196$

FRI: $d_4 = (.2)(0.196) + (.16) = 0.1992$

$\therefore$ $\boxed{19.92\ \%}$

4. Prove $\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)} \geq \frac{1}{2n}$, $n \geq 1$ by induction.

BC $\quad n=1 \quad \Rightarrow \quad \frac{1}{2} \geq \frac{1}{2\cdot1}$

IS $\quad$ Show: $\quad \frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots(2n)} \overset{*}{\geq} \frac{1}{2n}$

$$\Rightarrow \frac{1\cdot3\cdots(2n+1)}{2\cdot4\cdots(2n+2)} \geq \frac{1}{2n+2}, \text{ for } n \geq 1.$$

$$\frac{1\cdot3\cdots(2n+1)}{2\cdot4\cdots(2n+2)} \overset{*}{\geq} \frac{2n+1}{2n(2n+2)}$$

$$= \frac{2n}{2n(2n+2)} + \frac{1}{2n(2n+2)}$$

$$\geq \frac{1}{2n+2}$$

5. (a) Find a closed form expression for $1 + 3 + 5 + \cdots + (2n - 1)$.

(b) Make a conjecture about the terms of the following sequence, and prove your conjecture:

$$\frac{1}{3}, \frac{1+3}{5+7}, \frac{1+3+5}{7+9+11}, \ldots$$

(Note: This sequence was used by Galileo in his work on freely falling bodies.)

a) $1+3+\cdots+(2n-1) = \sum_{i=1}^{n} 2i-1 = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = 2\frac{(n+1)n}{2} - n = n^2$

b) $a_1 = \frac{1}{3}$

$a_2 = \frac{1+3}{5+7} = \frac{4}{12} = \frac{1}{3}$

$a_3 = \frac{1+3+5}{7+9+11} = \frac{9}{27} = \frac{1}{3}$

guess $a_n = 1/3$.

$$a_n = \frac{1+3+\cdots+(2n-1)}{(2n+1)+(2n+3)+\cdots+(2(2n)-1)}$$

$$= \frac{1+3+\cdots+(2n-1)}{1+3+\cdots+(2n-1)+(2n+1)+\cdots+(2(2n)-1) - (1+3+\cdots+(2n-1))}$$

$$= \frac{n^2}{(2n)^2 - n^2}$$

$$= \frac{n^2}{3n^2}$$

$$= \boxed{1/3}$$

6. If G is a connected simple planar graph with V vertices ($V \geq 3$) and E edges show: $E \leq 3V - 6$.

Planar $\Rightarrow$ $V - E + F = 2$

simple $\Rightarrow$ smallest cycle size $= 3$

$\Rightarrow$ $3F \leq$ boundaries $= 2E$

$\therefore 3F = 3(2 - V + E) = 6 - 3V + 3E \leq 2E$

$\Rightarrow$ $E \leq 3V - 6$

7. Decode the cipher text "CLRF" which was encoded using a hill cipher with

$$a = 1, b = 2, c = 0, d = 1$$

$\Rightarrow$ $E(x) \equiv x + 2y$

$E(y) \equiv y$ mod 26

$\Rightarrow$ $E(x) \equiv x + 2E(y)$ mod 26

$\Rightarrow$ $x \equiv E(x) - 2E(y)$ mod 26

$\therefore$ $D(x) \equiv x - 2y$

$D(y) \equiv y$ mod 26

$\therefore$ $D(2) \equiv 2 - 2(11) \equiv 2 + 4 \equiv 6$

$D(11) \equiv 11$

$D(17) \equiv 17 - 2(5) \equiv 7$

$D(5) \equiv 5$

$$\boxed{\begin{array}{c} G \\ L \\ H \\ F \end{array}}$$

8. Show that $5^n - 1$ is divisible by 4 for every positive integer n.

$$5^n - 1 \equiv 1^n - 1 \equiv 1 - 1 \equiv 0 \mod 4$$

9. Show that $n^3 - n$ is divisible by 3 for every positive integer n.

$$n^3 - n \equiv n(n^2-1) \equiv n(n+1)(n-1) \equiv 0 \mod 3$$

since $(n-1), n, (n+1)$ are 3 consecutive integers.

10. Show that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for every positive integer n.

$$\begin{aligned} 2^{n+2} + 3^{2n+1} &\equiv 2^{n+2} + 3 \cdot (3^2)^n \\ &\equiv 2^{n+2} + 3 \cdot 2^n \\ &\equiv 2^n(2^2+3) \\ &\equiv 2^n \cdot 7 \\ &\equiv 0 \mod 7 \end{aligned}$$

11. Show that $4^{2n+1} + 3^{n+2}$ is divisible by 13 for every positive integer n.

$$\begin{aligned} 4^{2n+1} + 3^{n+2} &\equiv 4 \cdot (4^2)^n + 3^{n+2} \\ &\equiv 4 \cdot 3^n + 3^{n+2} \\ &\equiv 3^n(4+3^2) \\ &\equiv 3^n \cdot 13 \\ &\equiv 0 \mod 13 \end{aligned}$$

4. Suppose a general wants to send a message to his troops behind enemy lines. He has several couriers that he can use to carry the message; however, up to three of them may be caught by the enemy. To ensure the message gets across, the general makes 4 copies of the message. To ensure the enemy doesn't intercept the entire message, the general cuts each of the four messages into five pieces and sends each courier with some combination of the different pieces. What is the minimum number of couriers the general needs to send so the message gets across and the enemy cannot intercept the entire message?

Let the message be cut into five parts A B C D E. If we can solve the problem with less than 20 couriers, one courier (by P.H.P.) must have at least 2 pieces. No courier can have 3 pieces since then we can not send the other 2 pieces on any couriers or else the enemy can get the entire message. Let the first courier carry A B. Then the four copies of C, D, and E must be sent separately.

We have:

COURIER - parts

1) A B
2) C A
3) C B
4) C B
5) C B
6) D
7) D
8) D
9) D
10) E
11) E
12) E
13) E

We are left to send A, A, A, B, B, and B. Without loss let courier 2) carry CA. Now we can not have the pair BD OR BE on any courier.

∴ B, B, B goes with courier 3), 4), 5) respectively. Now A can not go with D or E.

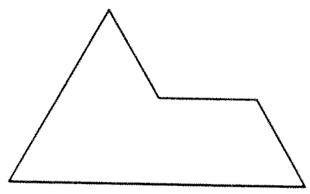
∴ 13 couriers is not enough.

14 COURIERS

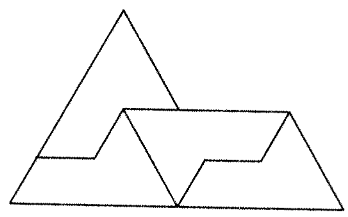
1) A B
2) A B
3) C A
4) C A
5) C B
6) C B
7) D
8) D
9) D
10) D
11) E
12) E
13) E
14) E

The minimum number of couriers is: $\boxed{14}$

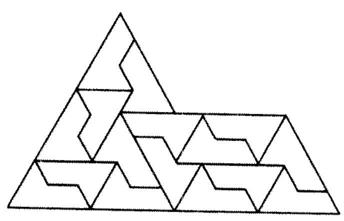
6. The following figure is called a left-facing sphinx.



Notice that a left-facing sphinx can be dissected into four smaller sphinxes: three that are right-facing sphinxes (once rotated) and one that is a left-facing sphinx.



A second dissection results in each of these new sphinxes becoming four smaller sphinxes in a similar way.



Let a_n denote the number of left-facing sphinxes after n such dissections into smaller sphinxes. That is, $a_0 = 1$, $a_1 = 1$, $a_2 = 10$, ...

Find a recurrence relation for a_n. You do not need to solve this recurrence. Include the initial condition in your answer.

Notice • After n dissections we have 4^n sphinxes

$\Rightarrow$ " " " " " $4^n - a_n$ right-facing sphinxes.

• A dissection of a right-facing sphinx gives 3 left-facing sphinxes.

• A dissection of a left-facing sphinx gives 1 left-facing sphinx.

$$\therefore \quad a_n = a_{n-1} + 3\left(4^{n-1} - a_{n-1}\right)$$

The recurrence relation is:

$$\boxed{\begin{array}{l} a_0 = 1 \\ a_n = 3 \cdot 4^{n-1} - 2a_{n-1} \end{array}}$$

7. At a party, everyone shakes hands with exactly three other people, except for one person, who shakes hands with only one person. What is the smallest number of people who can be at this party?

Consider the graph:

Vertices : people

edges : people who shook hands.

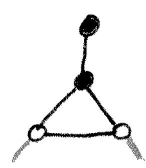
$\therefore$ Each vertex has an odd degree (1 or 3)

$\therefore$ There is an even # of vertices by the handshaking lemma.

- If there are 2 vertices we can have at most one edge :

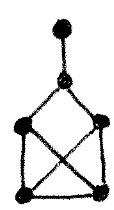
but one vertex must have degree 3.

- If there are 4 vertices we can have at most 4 edges :



but 3 vertices must have degree 3.

- With 6 vertices we may have :



The smallest number of people who can be at the party is: $\boxed{6}$

15. For all $n \geq 1$ show that $F_{3n} \equiv 0 \bmod 2$, where

$$F_1 = 1,$$
$$F_2 = 1,$$

$$F_n = F_{n-1} + F_{n-2}.$$

BC $(n=1)$

$\Rightarrow F_{3n} = F_3 = 2 \equiv 0 \mod 2$

IS Show: $F_{3n} \overset{*}{\equiv} 0 \mod 2$

$\Longrightarrow F_{3n+3} \equiv 0 \mod 2$, for $n \geq 1$.

$$
\begin{aligned}
F_{3n+3} &= F_{3n+2} + F_{3n+1} \\
&= F_{3n+1} + F_{3n} + F_{3n+1} \\
&= 2F_{3n+1} + F_{3n} \\
&\overset{*}{\equiv} 0 F_{3n+1} + 0 \\
&\equiv 0 \qquad \mod 2
\end{aligned}
$$