MATH 222

Final Practice Problems

1. For all $n \geq 0$, show that any deficient $(2^n \cdot 8) \times (2^n \cdot 14)$ board can be tiled with right trominoes.

2. Show that any deficient 25×10 board can be tiled with right trominoes.

3. If the price of gold does not change on a certain day, it will remain the same the next day with a probability of 0.6, go down with 0.2 and go up with 0.2. If it goes down on a certain day, it will continue to do so the next day with a probability of 0.3, stay put with 0.6 and go up with 0.1. If it goes up on a certain day, it will continue to do so the next day with a probability of 0.3, stay put with 0.6 and go down with 0.1. During a certain week, the price goes up on Monday. What is the probability that it will go down on Friday?

4. Prove $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \geq \frac{1}{2n}$, $n \geq 1$ by induction.

5. (a) Find a closed form expression for $1 + 3 + 5 + \cdots + (2n - 1)$.

(b) Make a conjecture about the terms of the following sequence, and prove your conjecture:

$$\frac{1}{3}, \frac{1+3}{5+7}, \frac{1+3+5}{7+9+11}, \cdots$$

(Note: This sequence was used by Galileo in his work on freely falling bodies.)

6. If G is a connected simple planar graph with V vertices $(V \geq 3)$ and E edges show: $E \leq 3V - 6$.

7. Decode the cipher text "CLRF" which was encoded using a hill cipher with

$$a = 1, b = 2, c = 0, d = 1$$

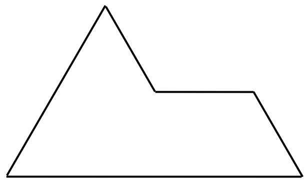
8. Show that $5^n - 1$ is divisible by 4 for every positive integer n.

9. Show that $n^3 - n$ is divisible by 3 for every positive integer n.

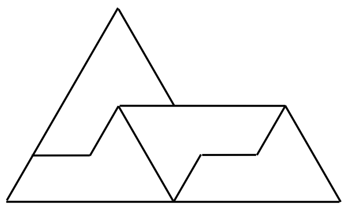
10. Show that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for every positive integer n.

11. Show that $4^{2n+1} + 3^{n+2}$ is divisible by 13 for every positive integer n.

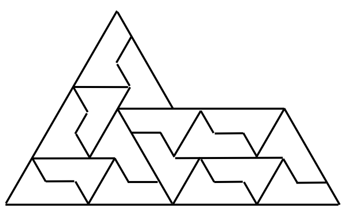
12. The following figure is called a left-facing sphinx.



Notice that a left-facing sphinx can be dissected into four smaller sphinxes: three that are right-facing sphinxes (once rotated) and one that is a left-facing sphinx.



A second dissection results in each of these new sphinxes becoming four smaller sphinxes in a similar way.



Let a_n denote the number of left-facing sphinxes after *n* such dissections into smaller sphinxes. That is, $a_0 = 1$, $a_1 = 1$, $a_2 = 10$, …

Find a recurrence relation for a_n .

13. Suppose a general wants to send a message to his troops behind enemy lines. He has several couriers that he can use to carry the message; however, up to three of them may be caught by the enemy. To ensure the message gets across, the general makes 4 copies of the message. To ensure the enemy doesn't intercept the entire message, the general cuts each of the four messages into five pieces and sends each courier with some combination of the different pieces. What is the minimum number of couriers the general needs to send so the message gets across and the enemy cannot intercept the entire message?

14. At a party, everyone shakes hands with exactly three other people, except for one person, who shakes hands with only one person. What is the smallest number of people who can be at this party?

15. For all $n \geq 1$ show that $F_{3n} \equiv 0 \bmod 2$, where

$$F_1 = 1,$$
$$F_2 = 1,$$

$$F_n = F_{n-1} + F_{n-2}.$$