Computing Science 204 (B1)
Properties of Trees
Tuesday, August 8

Theorem. Given a graph $G = (V, E)$ let $n = |V|$ be the number of vertices in G, and let $m = |E|$ be the number of edges in G. Given a vertex $v \in V$, the *degree* of v, denoted by $\deg(v)$ is the number of vertices in V that are adjacent to v. Recall that G is a *tree* if and only if it is connected and has no cycles. The following are true.

(a) $\sum_{v \in V} \deg(v) = 2\,m$
(b) G is connected if and only if it has a spanning tree.
(c) If G is a tree and v and w are any two distinct vertices in G, then there is a unique path joining v and w.
(d) G is a tree if and only if G is connected and $n = m + 1$.
(e) G is a tree if and only if G has no cycles and $n = m + 1$.
(f) If G is a tree with more than one vertex, then there are at least two vertices of degree 1.

proof.

(a) As we consider each edge $\{a, b\} \in E$ in the graph G, the edge contributes a count of 1 to each of $\deg(a)$ and $\deg(b)$, and therefore, a count of 2 to the sum $\sum_{v \in V} \deg(v)$. Thus, $2\,|E|$ accounts for $\deg(v)$, for all $v \in V$, and $\sum_{v \in V} \deg(v) = 2\,|E|$.

(b) If G has a spanning tree T, then T is a connected subgraph of G which has no cycles and which contains all the vertices of G. Therefore, G must be connected.

Conversely, if G is connected and G is not a tree, then G must contain a cycle. If we remove an edge from this cycle, then the resulting subgraph G_1 is still connected and has the same number of vertices as G. If G_1 is not a tree, then it must contain a cycle. If we remove an edge from this cycle, then the resulting subgraph G_2 is still connected and has the same number of vertices as G. If G_2 is not a tree, then it must contain a cycle. If we remove an edge from this cycle, then the resulting subgraph G_3 is still connected and has the same number of vertices as G. Since there are only finitely many edges in the graph, this process must eventually stop, say with a subgraph G_k of G which is still connected and has the same number of vertices as G, but this last subgraph G_k, has *no* cycles, and is therefore a spanning tree for G.

(c) If G is a tree and v and w are any two distinct vertices in G, there is a path joining v and w since G is connected. This path is unique, since if there were two different paths joining v and w, then G would contain a cycle.

(d) If G is a tree, then G is connected. We will show that $n = m + 1$ by induction on n. The result is true when $n = 1$, since the only possible tree has no edges in that case, and $m = 0$. Now suppose the result is true for all trees with k vertices, $1 \le k \le n$, and let $G = (V, E)$ be a tree with $|V| = n + 1$, and let $\{a, b\}$ be any edge of G. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are the trees obtained by removing the edge $\{a, b\}$ from G, we have
$$|V_1| + |V_2| = |V| \qquad \text{and} \qquad |E_1| + |E_2| = |E| - 1.$$

Applying the induction hypothesis to G_1 and G_2 we have

$$|E| = |E_1| + |E_2| + 1 = |V_1| - 1 + |V_2| - 1 + 1 = |V| - 1 = n - 1,$$

as required. Therefore the result is true for all positive integers n.

Conversely, suppose that G is connected and $|V| = |E| + 1$. Suppose that G contains a cycle C_r with r vertices and r edges. Since G is connected, the remaining vertices of G can each be connected to a vertex in C_r by a path in G. Each such connection requires at least one new edge. Therefore, in G, we must have $|E| \geq |V|$, which contradicts the assumption that $|V| = |E| + 1$. Therefore, G has no cycles, and is connected, thus, G is a tree.

(e) If G is a tree, then G has no cycles, and from part (d) G is connected and $|V| = |E| + 1$.

Conversely, suppose that G has no cycles and $|V| = |E| + 1$, suppose also that G is disconnected with components $G_1, G_2, \ldots, G_r$. For each i, with $1 \leq i \leq r$, select a vertex $v_i \in G_i$ and add the $r - 1$ edges

$$\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{r-1}, v_r\}$$

to G to obtain the graph $G' = (V, E')$. Then G' is a tree, and therefore from part (d) we have $|V| = |E'| + 1$. Thus, $|E'| + 1 = |V| = |E| + 1$, that is, $|E'| = |E|$, and so $r = 1$. Therefore, G is connected and $|V| = |E| + 1$, so by part (d), G is a tree.

(f) Let $G = (V, E)$ be a tree with $|V| \geq 2$, from part (d), we have $|E| = |V| - 1$, while from part (a), we have

$$2\big(|V| - 1\big) = 2\,|V| - 2 = \sum_{v \in V} \deg(v).$$

If $\deg(v) \geq 2$ for all $v \in V$, then

$$\sum_{v \in V} \deg(v) \geq 2\,|V|,$$

that is, $2\,|V| - 2 \geq 2\,|V|$, and this gives a contradiction unless there are two vertices v for which $\deg(v) < 2$. Since G is connected, there are no isolated vertices, that is, with degree 0, and therefore there must be two vertices with degree 1.

$\square$