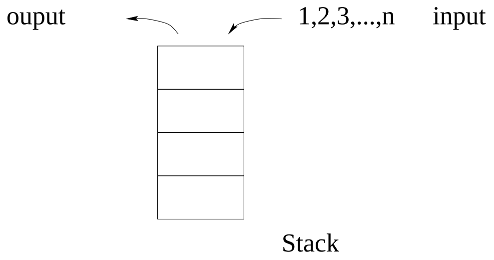
Computing Science 204 (B1)
Generating Permutations Using a Stack
Thursday, July 20

Let a_n be the number of ways to permute the integers 1, 2, 3, ..., n (or any list of n consecutive integers) using a stack.

Each integer in the input must be pushed onto the top of the stack in the order given. An entry may be popped from the top of the (nonempty) stack at any time. However, once an entry is popped from the stack, it may not be returned to either the top of the stack or the input left to be pushed onto the stack. The process continues until no entry is left on the stack.



(a) Calculate a_1, a_2, a_3, a_4.

(b) Show that for $n \geq 1$,

$$a_{n+1} = a_0 a_n + a_1 a_{n-1} + a_2 a_{n-2} + \cdots + a_{n-1} a_1 + a_n a_0.$$

where we define $a_0 = 1$.

(c) Solve this recurrence relation to find an explicit formula for a_n, $n \geq 1$.

SOLUTION:

(a) For $n = 1$, the output consists of the integer 1, which arises from the operations

$\{1\}$: `push 1, pop 1,`

therefore $a_1 = 1$.

For $n = 2$, the output consists of two permutations $1, 2$ and $2, 1$ which arise from the operations

$\{1, 2\}$: `push 1, pop 1, push 2, pop 2`
$\{2, 1\}$: `push 1, push 2, pop 2, pop 1,`

therefore $a_2 = 2$.

For $n = 3$, the output consists of the permutations arising from the operations

$\{1, 2, 3\}$: `push 1, pop 1, push 2, pop 2, push 3, pop 3`
$\{1, 3, 2\}$: `push 1, pop 1, push 2, push 3, pop 3, pop 2`
$\{2, 1, 3\}$: `push 1, push 2, pop 2, pop 1, push 3, pop 3`
$\{2, 3, 1\}$: `push 1, push 2, pop 2, push 3, pop 3, pop 1`
$\{3, 2, 1\}$: `push 1, push 2, push 3, pop 3, pop 2, pop 1,`

therefore $a_3 = 5$.

For $n = 4$, the output consists of the permutations arising from the operations

$\{1,2,3,4\}$:	`push 1, pop 1, push 2, pop 2, push 3, pop 3, push 4, pop 4`
$\{1,2,4,3\}$:	`push 1, pop 1, push 2, pop 2, push 3, push 4, pop 4, pop 3`
$\{1,3,2,4\}$:	`push 1, pop 1, push 2, push 3, pop 3, pop 2, push 4, pop 4`
$\{1,3,4,2\}$:	`push 1, pop 1, push 2, push 3, pop 3, push 4, pop 4, pop 2`
$\{1,4,3,2\}$:	`push 1, pop 1, push 2, push 3, push 4, pop 4, pop 3, pop 2`
$\{2,1,3,4\}$:	`push 1, push 2, pop 2, pop 1, push 3, pop 3, push 4, pop 4`
$\{2,1,4,3\}$:	`push 1, push 2, pop 2, pop 1, push 3, push 4, pop 4, pop 3`
$\{2,3,1,4\}$:	`push 1, push 2, pop 2, push 3, pop 3, pop 1, push 4, pop 4`
$\{3,2,1,4\}$:	`push 1, push 2, push 3, pop 3, pop 2, pop 1, push 4, pop 4`
$\{2,3,4,1\}$:	`push 1, push 2, pop 2, push 3, pop 3, push 4, pop 4, pop 1`
$\{2,4,3,1\}$:	`push 1, push 2, pop 2, push 3, push 4, pop 4, pop 3, pop 1`
$\{3,2,4,1\}$:	`push 1, push 2, push 3, pop 3, pop 2, push 4, pop 4, pop 1`
$\{3,4,2,1\}$:	`push 1, push 2, push 3, pop 3, push 4, pop 4, pop 2, pop 1`
$\{4,3,2,1\}$:	`push 1, push 2, push 3, push 4, pop 4, pop 3, pop 2, pop 1`

therefore $a_4 = 14$.

(b) The list of permutations above for $n = 4$ shows us how to count the number of possible permutations generated by the stack for the input $\{1, 2, \ldots, n, n+1\}$. The integer 1 can be in any of $n+1$ positions, for each such position, there are m_ℓ integers to the left of the 1, and m_r integers to the right of the 1, where $m_\ell + m_r = n$.

If $m_\ell \geq 1$, then the integers to the left of the 1 are just the integers $\{2, \ldots, m_\ell + 1\}$.

If $m_r \geq 1$, then the integers to the right of the 1 are just the integers $\{n + 1 - (m_r - 1), \ldots, n + 1\}$.

Therefore, the number of permutations that can be generated by the stack in the case when there are $m_\ell \geq 1$ integers to the left of the 1 and $m_r \geq 1$ integers to the right of the 1 is equal to $a_{m_\ell} \cdot a_{m_r}$. If $m_\ell = 0$ or $m_r = 0$, then the number of such permutations is a_n. Thus, if we define $a_0 = 1$, then

$$a_{n+1} = a_0 \cdot a_n + a_1 \cdot a_{n-1} + a_2 \cdot a_{n-2} + \cdots + a_{n-1} \cdot a_1 + a_n \cdot a_0,$$

that is, $a_{n+1} = \sum\limits_{k=0}^{n} a_k \cdot a_{n-k}$.

(c) Given a sequence $\{a_n\}_{n\geq 0}$, we define the *generating function* $F(x)$ of the sequence $\{a_n\}_{n\geq 0}$ to be the formal power series

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

that is, we don't worry about convergence. We can multiply two generating functions as follows, If $F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$ and $G(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$, then

$$\begin{aligned} F(x) \cdot G(x) &= \big(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots\big)\big(b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots\big) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \\ &\qquad + (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0) x^n + \cdots \end{aligned}$$

that is,

$$F(x) \cdot G(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.$$

This is called the *Cauchy product* of the two series.

Now note that

$$F(x)^2 = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n = \sum_{n=0}^{\infty} a_{n+1} x^n,$$

and therefore,

$$xF(x)^2 = \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = F(x) - a_0,$$

and since we defined $a_0 = 1$, then $xF(x)^2 = F(x) - 1$. We can solve this quadratic equation to get

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x},$$

and in order to determine which sign we should use, we have to look at the expansion of $\sqrt{1-4x}$ using the binomial theorem.

We have

$$\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} 4^n x^n,$$

where

$$\binom{\frac{1}{2}}{0} = 1$$

and for $n \geq 1$,

$$\begin{aligned}
\binom{\frac{1}{2}}{n} &= \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)}{n!} \\
&= \frac{1}{2^n \cdot n!} \cdot (1-2)(1-4)(1-6)\cdots(1-2n+2) \\
&= \frac{1}{2^n \cdot n!} \cdot (-1)(-3)(-5)\cdots(-(2n-3)) \\
&= \frac{(-1)^{n-1}}{2^n \cdot n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1)}{2n-1} \\
&= \frac{(-1)^{n-1}}{2^n \cdot n!(2n-1)} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (2n-3) \cdot (2n-2) \cdot (2n-1) \cdot (2n)}{2 \cdot 4 \cdot 6 \cdots 2n} \\
&= \frac{(-1)^{n-1}(2n)!}{2^n \cdot n!(2n-1) \cdot 2^n \cdot n!} \\
&= \frac{(-1)^{n-1}}{4^n} \frac{1}{2n-1} \binom{2n}{n}
\end{aligned}$$

so that

$$\sqrt{1-4x} = 1 - \sum_{n=1}^{\infty} \frac{1}{2n-1} \binom{2n}{n} x^n.$$

Thus, we have to take the minus sign in the solution of the quadratic, otherwise a_n would be negative for $n \geq 1$. Therefore, we have

$$F(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2n-1} \binom{2n}{n} x^{n-1} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2n+1} \binom{2n+2}{n+1} x^n,$$

and finally,

$$a_n = \frac{1}{2} \frac{1}{2n+1} \binom{2n+2}{n+1} = \frac{1}{n+1} \binom{2n}{n}$$

for $n \geq 0$, that is, a_n is the n^{th} *Catalan Number.*