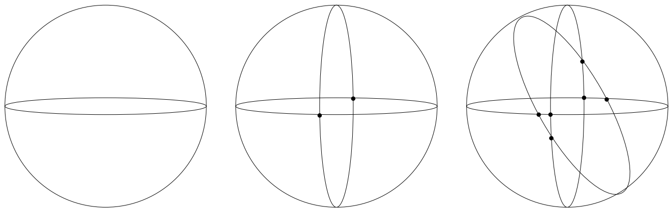
Computing Science 204 (B1)
Decomposition of a Sphere by Great Circles
Tuesday, July 25

Example. Into how many regions do n great circles (no three concurrent) decompose the surface of the sphere on which they lie?

Solution. For $n \geq 0$, let a_n be the number of regions on the surface of a sphere formed by n great circles of which no three are concurrent. From the figure below it is obvious that $a_0 = 1$, $a_1 = 2$, $a_2 = 4$ and $a_3 = 8$.



Thus, it looks at first like $a_n = 2^n$ for all $n \geq 0$, but this is not the case.

Suppose that n circles have been drawn on the surface of the sphere so that no three are concurrent, and suppose that an $(n+1)^{\text{st}}$ circle is added so that no three of the $n+1$ circles are concurrent.

The new circle meets each of the old circles in two points, making $2n$ points of intersection on the new circle, and these $2n$ points are all different since no three of the circles are concurrent.

The $2n$ points divide the new circle into $2n$ arcs. Each of these arcs divides one of the existing regions into two parts, so there are

$$a_n + 2n$$

regions formed by the $n+1$ great circles. Therefore, a_n satisfies the discrete initial value problem

$$\begin{aligned} a_{n+1} &= a_n + 2n, \quad n \geq 1 \\ a_1 &= 2. \end{aligned}$$

The characteristic equation of the homogeneous equation

$$a_{n+1} - a_n = 0$$

is $\lambda - 1 = 0$, and the general solution to the homogeneous equation is

$$h_n = A \cdot 1^n = A, \quad n \geq 1$$

where A is an arbitrary constant.

A particular solution to the nonhomogeneous equation

$$a_{n+1} - a_n = 2n, \quad n \geq 1$$

is

$$p_n = n(n-1), \quad n \geq 1$$

so that the general solution to the nonhomogeneous equation is

$$a_n = h_n + p_n = A + n(n-1), \quad n \geq 1$$

Applying the initial condition $a_1 = 2$, we find $A = 2$, so that the number of regions into which the surface of a sphere is decomposed by n great circles, no three of which are concurrent, is

$$a_n = n(n-1) + 2, \quad n \geq 1,$$

not 2^n.