CMPUT 204 (B1) – Assignment #1

DUE DATE: July 19, 2000

Solutions

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1. *Insertion Sort*

A recursive version of the sorting algorithm *InsertionSort* is given below.

```
void InsertionSort(int *L, int m, int n)
{
    if (m > 0)
    {
        int j, k = m, temp = L[m-1];
        while ((k < n) && (L[k] < temp))
            k++;
        j = k-1;
        for (k = m; k <= j; k++)
            L[k-1] = L[k];
        L[j] = temp;
        InsertionSort(L,m-1,n);
    }
}
```

In order to sort a list `L` with `n` entries, the procedure is called as `InsertionSort(L,n,n)`.

(a) Explain briefly how the algorithm works.

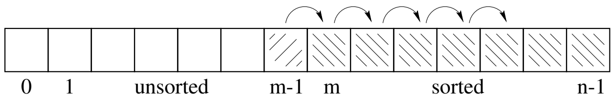
(b) What is a worst–case input for the algorithm? That is, for what input would the algorithm make the most number of comparisons of list entries?

(c) Let $T(n)$ be the number of comparisons of list entries made in the worst case when *InsertionSort* is performed on a list L containing n entries. Find a recurrence relation together with an initial condition satisfied by $T(n)$ for $n \geq 1$.

(d) Solve this recurrence relation to get $T(n)$ for $n \geq 1$.

SOLUTION:

(a) For a fixed `m > 0`, the algorithm assumes that the portion of the array from `L[m]` to `L[n-1]` has been sorted, and then compares the entry in `L[m-1]` with each entry in the sorted portion until it finds a place to insert it. If it is to be inserted at entry `L[j]`, then the entries from `L[m]` to `L[j]` are shifted to the left one position to leave the entry `L[j]` vacant so that the entry `L[m-1]` may be inserted.



After the entry has been inserted in its correct place, then the portion of the array from `L[m-1]` to `L[n-1]` has been sorted, and the algorithm calls itself again with parameter `m-1`. The last call for which any comparisons are made occurs when `m-1 = 0`, that is, `m = 1`, and the algorithm terminates with the next call, that is, when `m = 0`.

(b) Obviously, because of the way the algorithm works, a worst–case input, that is, an input for which the largest number of comparisons are done, consists of a list that is already sorted, but in the reverse order.

(c) As noted in class, the algorithm really depends on two parameters, m and n, even though n doesn't change during the execution of the algorithm, and therefore the number of comparisons done depends on both m and n.

If we let $a(m,n)$ be the number of comparisons made by the algorithm when it is called with parameter values m and n, then it is clear that

$$a(m,n) = n - m + a(m-1,n)$$

for $1 \leq m \leq n$, and that $a(0,n) = 0$.

(d) Note that

$$\begin{aligned} a(n,n) &= 0 + a(n-1,n) \\ &= 0 + 1 + a(n-2,n) \\ &= 0 + 1 + 2 + a(n-3,n) \\ &= 0 + 1 + 2 + 3 + a(n-4,n) \\ &\vdots \\ &= 0 + 1 + 2 + \cdots + n - 1 + a(0,n), \end{aligned}$$

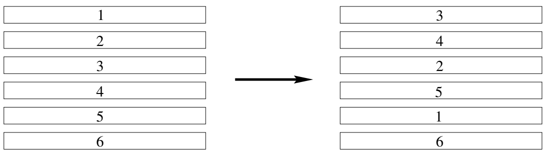
that is, $T(n) = a(n,n) = 1 + 2 + 3 + \cdots + n - 1 = \dfrac{n(n-1)}{2}$.

2. *Card Shuffling*

(a) Given an ordinary deck of playing cards, arranged in any order, how many shuffles are required before the deck returns to its original order? To idealize the real life situation, we stipulate that only the *modified perfect faro shuffle* is permitted, that is, cut the deck into two equal 26-card packs; then proceed by alternating cards from each pack. The cards formerly in positions 1, 2, ..., 26 are moved to positions 2, 4, ..., 52; and the cards formerly in positions 27, 28, ..., 52 are moved to positions 1, 3, ..., 51. Thus, if a particular card starts in position x, it will end in position y, where $1 \le y \le 52$ and $2x \equiv y \mod 53$. After n shuffles, this particular card will be in position w, where $1 \le w \le 52$ and $2^n x \equiv w \mod 53$.

More generally, if we have a deck of m cards, where m is an even integer, find the smallest integer n such that n shuffles will return the cards to the original order. What is n for $m = 6$? $m = 8$? $m = 12$? $m = 62$?

(b) Suppose that instead of performing a modified perfect faro shuffle as described above, we shuffle as follows: Take the bottom and top cards of the deck and place them on the table to start a new deck. Then take the remaining bottom and top cards and place them on the newly started pile. Continue this process until all cards are gone from the original deck. For example, if the original deck has six cards, then we shuffle them as shown in the figure.



Show that the cards in a deck of 2^n cards will return to their original positions after $n+1$ such shuffles.

SOLUTION:

(a) To determine the number of shuffles necessary to return the cards to their original order, we have to find an integer n such that

$$2^n x \equiv x \mod 53$$

for every x such that $1 \le x \le 52$. Since 53 is a prime and $\gcd(x, 53) = 1$ for each such x, then we may cancel the x from both sides of the congruence; so we have to solve the congruence

$$2^n \equiv 1 \mod 53.$$

However, from Fermat's Little Theorem, we know that

$$2^{52} \equiv 1 \mod 53,$$

and therefore, the cards will return to their original order after 52 shuffles, and it can be shown that this is the least number of shuffles required.

More generally, if we have a deck of m cards, where m is even, then n shuffles will return the cards to their original order provided that

$$2^n \equiv 1 \mod m+1.$$

since in this case, $2^n x \equiv x \mod m+1$ for all x such that $1 \leq x \leq m$.

If $m=6$, then we need only $n=3$ shuffles, since $2^3 \equiv 1 \mod 7$.

If $m=8$, then we need only $n=6$ shuffles, since $2^6 \equiv 1 \mod 9$.

If $m=12$, then we need only $n=12$ shuffles, since $2^{12} \equiv 1 \mod 13$.

If $m=62$, then we need only $n=6$ shuffles, since $2^6 \equiv 1 \mod 63$.

(b) This type of shuffle was first investigated by Gaspard Monge, and is usually known as *Monge's shuffle*.

Given a deck of $2N$ cards, it is easy to check that after one shuffle the card in position x goes to position y, where

$$y \equiv \pm 2x \mod 4N+1.$$

In fact, cards in position x, where $1 \leq x \leq N$, go to position $-2x \mod 4N+1$, while cards in position x where $N+1 \leq x \leq 2N$, go to position $+2x \mod 4N+1$.

Therefore, the number of shuffles required to restore the original order is the smallest positive integer s for which

$$2^s \equiv \pm 1 \mod 4N+1,$$

that is, we want the smallest s for which either $2^s \equiv +1 \mod 4N+1$ or $2^s \equiv -1 \mod 4N+1$.

In our case, $2N = 2^n$, and $4N = 2^{n+1}$, so we want the smallest s such that

$$2^s \equiv -1 \mod 2^{n+1}+1,$$

and this is $s = n+1$, since

$$2^{n+1} \equiv -1 \mod 2^{n+1}+1.$$

Therefore, with a deck of 2^n cards, the deck is returned to its original order after exactly $n+1$ Monge shuffles.

3. *Monotone Subsequences.*

(a) Given a sequence $a_1,\ a_2,\ a_3,\ \dots\ a_{n^2+1}$ of any n^2+1 different positive integers, show that there is either an increasing subsequence of $n+1$ terms, or else there is a decreasing sequence of $n+1$ terms.

For example, if $n = 3$, then any sequence of 10 different positive integers either contains an increasing sequence of four terms or else a decreasing sequence of four terms.

(b) Show that the result in part (a) is the best possible in the sense that it is not true for any shorter original sequence, that is, if we start with only n^2 different positive integers, it is possible to have no increasing or decreasing sequence of $n+1$ terms.

SOLUTION:

(a) Let the sequence be $a_1,\ a_2,\ a_3,\ \dots\ a_{n^2+1}$, and associate with each term a_k the positive integer t_k which gives the length of the longest increasing subsequence which starts at a_k. Thus, there are n^2+1 integers t_k.

Now, if any of the t_k's are $n+1$ or larger, then we have found an increasing subsequence of length at least $n+1$.

On the other hand, if all of the t_k's are less than $n+1$, then each t_k has a value between 1 and n. Therefore, we have n^2+1 pigeons (the t_k's) which we want to put into n pigeonholes (the values of the t_k's).

From Dirichlet's pigeonhole principle, one of these pigeonholes must contain at least

$$\left\lfloor \frac{(n^2+1)-1}{n} \right\rfloor + 1 = n+1$$

pigeons, that is, at least $n+1$ of the t_k's must be equal. Now we will show that the a_k's associated with these equal t_k's must form a decreasing subsequence.

If $i < j$ and a_i and a_j have equal t's, then we must have $a_i > a_j$. Otherwise, if $a_i \le a_j$, and we append a_i onto the front of the increasing subsequence starting at a_j, then we form an increasing subsequence starting at a_i which has length $t_j + 1 = t_i + 1 > t_i$. However, this contradicts the definition of t_i, therefore the a_k's associated with the equal t_k's form a decreasing subsequence of length $n+1$.

(b) In order to show that this is the best result possible, we give a sequence of n^2 terms which has no increasing subsequence of length $n+1$ and which also has no decreasing subsequence of length $n+1$. The sequence is as follows:

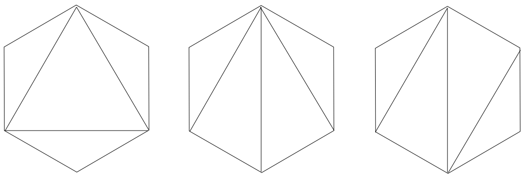
$$n, n-1, \dots, 1, 2n, 2n-1, \dots, n+1, 3n, 3n-1, \dots, 2n+1, \dots, n^2, n^2-1, \dots, (n-1)n+1.$$

Here the numbers from 1 to n^2 are arranged in a pattern such that the longest increasing subsequences are all n terms long, and the longest decreasing subsequences are also all n terms long.

Note that if we try to insert the number n^2+1 anywhere into this sequence, we will create either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

4. *Number of Triangulations of a Convex n-gon.*

Let a_n be the number of ways that a convex polygon with n sides can be divided into triangles by drawing diagonals that do not intersect. (A polygon is *convex* if all its diagonals lie in the interior of the polygon.)



(a) Calculate $a_3,\ a_4,\ a_5,\ a_6$.

(b) Show that no matter how a convex polygon with n sides is triangulated, the number of diagonals is always $n-3$ and the number of triangles is always $n-2$.

(c) Define $a_2 = 1$, and show that for $n \geq 3$,

$$a_n = a_2\, a_{n-1} + a_3\, a_{n-2} + a_4\, a_{n-3} + \cdots + a_{n-1}\, a_2$$

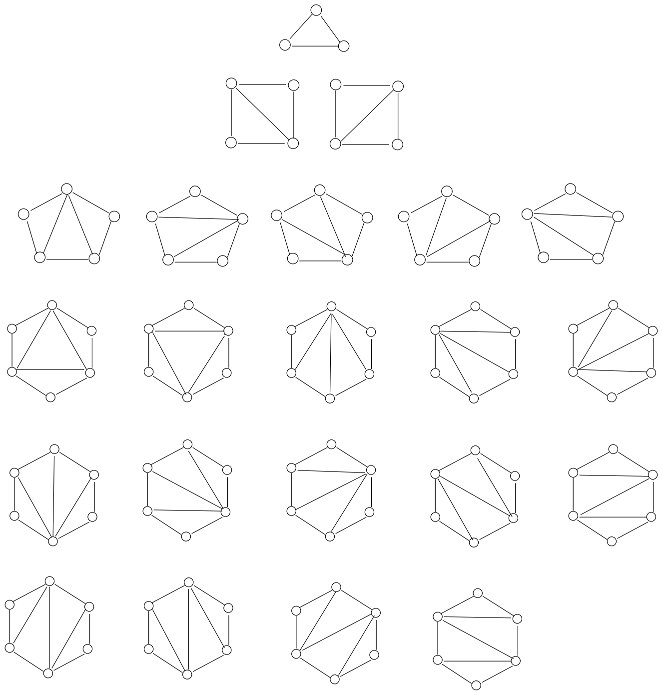
(d) Show that the sequence $\{a_n\}_{n\geq 3}$ also satisfies the discrete initial value problem

$$\begin{aligned}(n-3)a_n &= \frac{n}{2}\left(a_3\, a_{n-1} + a_4\, a_{n-2} + \cdots + a_{n-1}\, a_3\right), \quad n \geq 4\\ a_3 &= 1.\end{aligned}$$

(e) Solve these recurrence relations to find an explicit formula for $a_n,\ n \geq 3$.

SOLUTION:

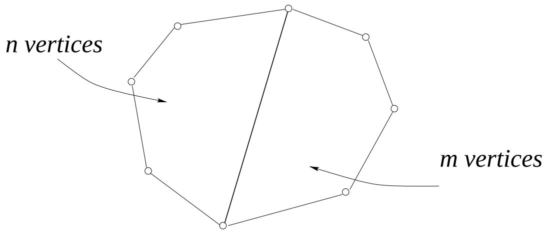
(a) We can see from the figure below that $a_3 = 1,\ a_4 = 2,\ a_5 = 5,$ and $a_6 = 14$.



(b) We will show that no matter how a convex polygon with n sides is triangulated, there are always $n-3$ diagonals and $n-2$ triangles. The proof is by induction on the number of vertices k.

When $k=4$, then from the figure above, the number of diagonals needed is $k-3=1$ and the number of triangles formed is $k-2=2$.

Now suppose that the result is true for all convex poygons with fewer than k vertices, and consider a polygon with k vertices and divide it into two nonoverlapping convex polygons with n vertices and m vertices, where $k=n+m-2$.



By the induction hypothesis, we can triangulate the convex n–gon with $n-3$ diagonals to form $n-2$ triangles, and we can triangulate the convex m–gon with $m-3$ diagonals to form $m-2$ triangles. Therefore, the convex k–gon can be triangulated with

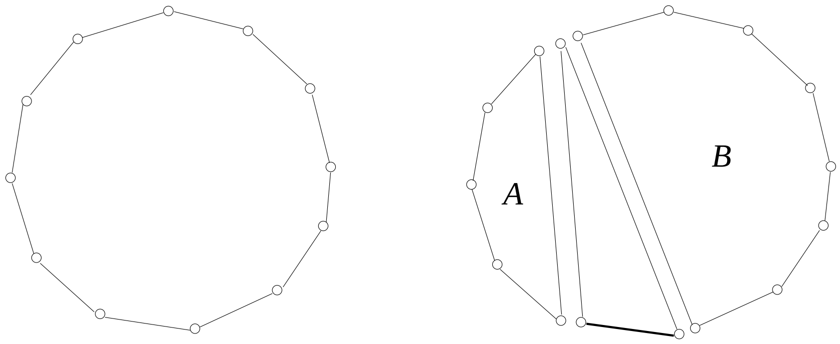
$$n-3+m-3+1=n+m-2-3=k-3$$

diagonals (this includes the edge shared by the n–gon and the m–gon), into

$$n-2+m-2=n+m-2-2=k-2$$

triangles. Therefore, by the principle of mathematical induction, the result is true for all convex n-gons.

(c) Let a_n, $n \geq 3$, be the number of ways of dividing a convex polygon with n vertices into triangles by drawing nonintersecting diagonals, and define $a_2=1$. Consider an convex n–gon and single out one side. Joining the ends of this rooted side with one of the other $n-2$ vertices produces a triangle and two other component figures, A and B, as shown below.



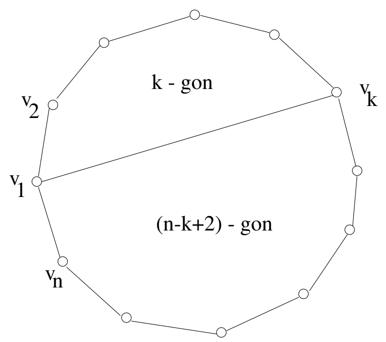
Let component A have k sides, then component B has $n-k+1$ sides. The two diagonals drawn are duplicated in the base triangle and the two components, so there are a total of $n+4$ edges in the three components.

The number of ways to triangulate component A is a_k and the number of ways to triangulate component B is a_{n-k+1}, so there are $a_k\, a_{n-k+1}$ ways to triangulate the original n–gon with the particular base triangle as the partition between the components. Since every contribution to a_n contains one triangle on the rooted side, and there are $n-2$ possible partitioning base triangles, the various possibilities correspond to values of k (the number of sides of A) from 2 to $n-1$. so that

$$a_n=\sum_{k=2}^{n-1} a_k\, a_{n-k+1}=a_2\, a_{n-1}+a_3\, a_{n-2}+a_4\, a_{n-3}+\cdots+a_{n-1}\, a_2$$

for $n \geq 3$.

(d) Now consider a convex n-gon with vertices $v_1, v_2, \ldots, v_n$, and single out one diagonal, say $v_1 v_k$. This diagonal divides the n-gon into two convex polygons:



a k-gon on top whose vertices are $v_1, v_2, \ldots, v_k$, and an $(n-k+2)$-gon on the bottom whose vertices are $v_k, v_{k+1}, \ldots, v_n, v_1$.

Reasoning as in part (c), we find the number of triangulations which include the diagonal $v_1 v_k$ is $a_k a_{n-k+2}$. There are $n-2$ possible diagonals coming from vertex v_1, namely, $v_1 v_3, v_1 v_4, \ldots, v_1 v_{n-1}$, so the total number of triangulations corresponding to v_1 is

$$a_3 a_{n-1} + a_4 a_{n-2} + \cdots + a_{n-1} a_3.$$

By symmetry, this sum is the same for any of the other n vertices v_i. This means that

$$n\left(a_3 a_{n-1} + a_4 a_{n-2} + \cdots + a_{n-1} a_3\right)$$

counts, for every possible diagonal, the total number of triangulations using that diagonal, two times each, once for each vertex on the diagonal. Therefore, the expression

$$\frac{n}{2}\left(a_3 a_{n-1} + a_4 a_{n-2} + \cdots + a_{n-1} a_3\right)$$

counts, for every possible diagonal, the total number of triangulations using that diagonal.

Now, since every triangulation uses $n-3$ diagonals, this expression counts each triangulation of the n-gon exactly $n-3$ times. Therefore, we must have

$$(n-3)a_n = \frac{n}{2}\left(a_3 a_{n-1} + a_4 a_{n-2} + \cdots + a_{n-1} a_3\right).$$

(e) From part (a) we have

$$a_{n+1} = a_2 a_n + a_3 a_{n-1} + a_4 a_{n-2} + \cdots + a_{n-1} a_3 + a_n a_2,$$

and since we defined $a_2 = 1$, then

$$a_{n+1} - 2a_n = a_3 a_{n-1} + a_4 a_{n-2} + \cdots + a_{n-2} a_4 + a_{n-1} a_3,$$

so that

$$\frac{n}{2}\left(a_{n+1} - 2a_n\right) = \frac{n}{2}\left(a_3 a_{n-1} + a_4 a_{n-2} + \cdots + a_{n-2} a_4 + a_{n-1} a_3\right) = (n-3)a_n$$

from part (d). Therefore, the sequence $\{a_n\}_{n \geq 2}$ satisfies the discrete initial value problem

$$\begin{aligned} n a_{n+1} &= (4n-6)a_n, \quad n \geq 3 \\ a_2 &= 1, \end{aligned}$$

and we solve this by making the substitution $b_{n+1} = n a_{n+1}$, and note first that $b_2 = 1$.

next, note that

$$b_{n+1} = na_{n+1} = (4n-6)\frac{b_n}{n-1},$$

so that

$$\frac{b_{n+1}}{b_n} = \frac{2(2n-3)}{n-1} = \frac{(2n-2)(2n-3)}{(n-1)(n-1)},$$

and therefore,

$$b_{n+1} = \frac{b_{n+1}}{b_n} \cdot \frac{b_n}{b_{n-1}} \cdot \frac{b_{n-1}}{b_{n-2}} \cdots \frac{b_3}{b_2},$$

where are the last stage, we used the fact that $b_2 = 1$. Thus,

$$b_{n+1} = \frac{(2n-2)(2n-3)}{(n-1)(n-1)} \cdot \frac{(2n-4)(2n-5)}{(n-2)(n-2)} \cdots \frac{2\cdot 1}{1 \cdot 1} = \frac{(2n-2)!}{(n-1)!(n-1)!} = \binom{2n-2}{n-1},$$

and

$$a_{n+1} = \frac{b_{n+1}}{n} = \frac{1}{n}\binom{2n-2}{n-1}.$$

The number of triangulations of a convex n-gon by nonintersecting diagonals is therefore

$$a_n = \frac{1}{n-1}\binom{2n-4}{n-2}$$

for $n \geq 3$.

The integer

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is called the n^{th} *Catalan number* after the mathematician Eugene Charles Catalan (1814–1894). The number of triangulations of a convex n-gon was first counted by Euler, and the Catalan numbers were rediscovered by Catalan in connection with another problem.

5. *A Familiar Sequence.*

Given the discrete initial value problem

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + na_0, \quad n \geq 1 \\ a_0 &= 1, \end{aligned}$$

(a) Calculate the values of a_n for small values of n; from the observed pattern, make a conjecture as to the general term a_n.

(b) Find the generating function

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

for the sequence $\{a_n\}$, and use this generating function to prove the conjecture in part (a).

SOLUTION:

(a) Note that

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 1 \cdot a_0 = 1 \cdot 1 = 1 \\ a_2 &= a_1 + 2 \cdot a_0 = 1 + 2 \cdot 1 = 3 \\ a_3 &= a_2 + 2 \cdot a_1 + 3 \cdot a_0 = 3 + 2 \cdot 1 + 3 \cdot 1 = 8 \\ a_4 &= a_3 + 2 \cdot a_2 + 3 \cdot a_1 + 4 \cdot a_0 = 8 + 2 \cdot 3 + 3 \cdot 1 + 4 \cdot 1 = 21 \\ &\vdots \end{aligned}$$

and it appears that for all $n \geq 1$, we have $a_n = F_{2n}$, that is, the Fibonacci numbers with even indices.

(b) Let

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}_{n \geq 0}$, and let

$$G(x) = \sum_{n=0}^{\infty} n x^n$$

be the generating function for the sequence $\{n\}_{n \geq 0}$, then the Cauchy product of the two series is given by

$$F(x) \cdot G(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = \sum\limits_{k=0}^{n} k a_{n-k} = a_{n-1} + 2a_{n-2} + \cdots + na_0 = a_n$ for $n \geq 1$, while $c_0 = 0$. Therefore, since $a_0 = 1$, we have

$$F(x) = 1 + F(x) \cdot G(x).$$

However, the geometric series converges for $-1 < x < 1$, so that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{and differentiating} \qquad \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} n x^n,$$

for $-1 < x < 1$, that is, $G(x) = \dfrac{x}{(1-x)^2}$.

Therefore,

$$F(x) = 1 + \frac{xF(x)}{(1-x)^2},$$

so that

$$F(x)\left[1 - \frac{x}{(1-x)^2}\right] = 1,$$

that is,

$$F(x)\left[\frac{1-3x+x^2}{1-2x+x^2}\right] = 1,$$

and

$$F(x) = \frac{1-2x+x^2}{1-3x+x^2} = 1 + \frac{x}{1-3x+x^2}.$$

But this is the generating function for the sequence whose n^{th} term is F_{2n} for $n \geq 1$, and 1 for $n = 0$, and since generating functions are unique, we must have

$$a_n = \begin{cases} F_{2n} & \text{for} \quad n \geq 1, \\ 1 & \text{for} \quad n = 0, \end{cases}$$

which agrees with the results in part (a).