Computing Science 204 (B1)

Probability of No Consecutive Heads

Wednesday, July 12

Let p_n be the probability that in n tosses of a fair coin two consecutive heads do not appear.

(a) We will show that p_n satisfies the *second order homogeneous linear difference equation with constant coefficients* given by

$$p_{n+2} = \frac{1}{2}\, p_{n+1} + \frac{1}{4}\, p_n$$

for $n \geq 1$, together with the *initial conditions*

$$p_1 = 1 \qquad p_2 = \frac{3}{4}.$$

(b) Assuming a solution of the form $p_n = \alpha^n$ for some constant α, we find the *auxiliary equation* for the homogeneous equation.

(c) If α_1 and α_2 are the *distinct real roots* of the auxiliary equation, the *general solution* to the homogeneous equation can be written in the form

$$p_n = A\,\alpha_1^n + B\,\alpha_2^n$$

where A and B are *arbitrary constants,* (independent of n). We use the initial conditions to determine the constants A and B.

(d) Let $a_n = 2^n\, p_n$, we show that

$$a_{n+2} = a_{n+1} + a_n$$

for all $n \geq 1$, where $a_1 = 2$, and $a_2 = 3$; so that the sequence $\{a_n\}_{n\geq 1}$ is part of the **Fibonacci sequence**:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\quad \cdots$$

SOLUTION:

(a) For each $n \geq 1$, let a_n be the number of sequences of H's and T's of length n which contain no consecutive H's. Any sequence of H's and T's of length $n+2$ which contains no consecutive H's either starts with an H or with a T; there are exactly a_{n+1} such sequences that start with a T, and exactly a_n such sequences that start with an H. Therefore, $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$, and clearly, $a_1 = 2$ while $a_2 = 3$. Since $p_n = \dfrac{a_n}{2^n}$ for $n \geq 1$, then $\{p_n\}_{n\geq 1}$ satisfies the difference equation

$$p_{n+2} = \frac{1}{2}\, p_{n+1} + \frac{1}{4}\, p_n \tag{$*$}$$

for $n \geq 1$, together with the initial conditions

$$p_1 = 1 \qquad p_2 = \frac{3}{4}. \tag{$**$}$$

(b) Substituting $p_n = \alpha^n$ into $(*)$, we get the quadratic equation

$$\alpha^2 = \frac{1}{2}\alpha + \frac{1}{4},$$

this is the auxiliary equation for the homogeneous euqation $(*)$.

(c) The distinct real roots of the auxiliary equation are given by

$$\alpha_1 = \frac{1+\sqrt{5}}{4} \qquad \text{and} \qquad \alpha_2 = \frac{1-\sqrt{5}}{4}$$

so that the general solution to $(*)$ can be written in the form $p_n = A\alpha_1^n + B\alpha_2^n$. Using the initial conditions $(**)$ the constants A and B must satisfy the linear equations

$$\alpha_1 A + \alpha_2 B = 1$$

$$\alpha_1^2 A + \alpha_2^2 B = \frac{3}{4},$$

whose unique solution is given by

$$A = \frac{\frac{3}{4} - \alpha_2}{\alpha_1(\alpha_1 - \alpha_2)} = \frac{4}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{4}\right)^2$$

and

$$B = \frac{\alpha_1 - \frac{3}{4}}{\alpha_2(\alpha_1 - \alpha_2)} = -\frac{4}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{4}\right)^2.$$

Therefore,

$$p_n = \frac{4}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{4}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{4}\right)^{n+2}\right].$$

(d) We have already seen that if $a_n = 2^n\, p_n$, then

$$a_{n+2} = a_{n+1} + a_n$$

for all $n \geq 1$, where $a_1 = 2$ and $a_2 = 3$; so that the sequence $\{a_n\}_{n\geq 1}$ is part of the Fibonacci sequence:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\quad \cdots$$