**Computing Science 204 (B1)**
**Asymptotic Growth Rate of Powers and Exponentials**
Tuesday, July 18

**Example 1.**

Let $g(n) = 2^n$ for $n \geq 1$, we will show that given any real number $\alpha > 0$, no matter how large,

$$n^\alpha \in O(2^n) \qquad \text{but} \qquad n^\alpha \notin \Omega(2^n).$$

**proof.** From L'Hôpital's rule, we see that

$$\lim_{n\to\infty} \frac{n^\alpha}{2^n} = 0$$

for all $\alpha > 0$, and therefore there exists an $n_0 \geq 1$ such that

$$0 < \frac{n^\alpha}{2^n} < 1$$

for all $n \geq n_0$. Thus, $n^\alpha \in O(2^n)$ for all $\alpha > 0$. On the other hand, suppose there exists an $\alpha > 0$ for which $n^\alpha \in \Omega(2^n)$, then there would exist a $c > 0$ and an integer $n_0 \geq 1$, such that

$$n^\alpha \geq c\, 2^n$$

for all $n \geq n_0$; that is

$$\frac{n^\alpha}{2^n} \geq c$$

for all $n \geq n_0$. Letting $n \to \infty$ in this inequality, we get $c \leq 0$, which contradicts the fact that $c > 0$. Therefore, $n^\alpha \notin \Omega(2^n)$ for any $\alpha > 0$.

$\square$

**Example 2.**

Let $a$ and $b$ be real numbers satisfying $b > a > 1$.

(a) Show that $\log_a n \in \Theta(\log_b n)$;

(b) Order the functions $f(n) = a^n$ and $g(n) = b^n,\ n \geq 1$, by their asymptotic growth rate.

**proof.**

(a) Since $a^{\log_a n} = n$, taking the logarithm base $b$ of both sides of this equality, we have

$$\log_b \left(a^{\log_a n}\right) = \log_a n \cdot \log_b a = \log_b n,$$

so that $\log_b n = \log_b a \cdot \log_a n$. Now, since $b > a > 1$, then $0 < \log_b a < 1$, and taking $c = \log_b a$, we have

$$\log_b n \leq c \cdot \log_a n$$

for all $n \geq 1$. Thus, $\log_b \in O(\log_a n)$.

Also, since

$$\log_b n \geq c \cdot \log_a n$$

for all $n \geq 1$, then $\log_b n \in \Omega(\log_a n)$.

Therefore, $\log_b n \in \Theta(\log_a n) = O(\log_a n) \cap \Omega(\log_a n)$.

(b) Since $b > a > 1$, then $0 < \dfrac{a}{b} < 1$, so that

$$\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n = 0.$$

Therefore, there exists an integer $n_0 \in \mathbb{N}$ such that

$$0 < \frac{a^n}{b^n} < 1$$

for all $n \geq n_0$, so that $a^n \in O(b^n)$. However, since

$$\lim_{n\to\infty} \frac{b^n}{a^n} = \lim_{n\to\infty} \left(\frac{b}{a}\right)^n = +\infty,$$

then $a^n \notin \Omega(b^n)$.

$\square$