Computing Science 204 (B1)
Dirichlet's Pigeonhole Principle
Wednesday, July 12

The pigeonhole principle states that if $n+1$ pigeons occupy n pigeonholes, then at least one pigeonhole will contain at least two pigeons. It may be that all $n+1$ pigeons are in the same pigeonhole, or every pigeonhole might have exactly three pigeons, but in any case, at least one pigeonhole will contain more than one pigeon.

Example. If $n+1$ integers are selected from the set $\{1, 2, \ldots, 2n\}$, then there exist two integers a and b from this selection such that either a divides b or b divides a.

proof. We can write each of the $n+1$ numbers selected uniquely as a power of 2 times an odd integer, for example, $48 = 2^4 \cdot 3$.

There are only n different odd integers possible in this representation of the integers $1, 2, \ldots, 2n$. Since we have selected $n+1$ numbers, at least two of them will have the same odd parts. The smaller power of 2 will divide the larger power of 2 evenly, so these two numbers satisfy the requirements.

$\square$

Note. The result in the example above is called a "best possible result" because if we relax the conditions of the hypothesis even slightly, the result is no longer true. For example, if we selected only n integers from the set $\{1, 2, \ldots, 2n\}$, we might not have a pair that divide evenly. In fact, none of the n integers

$$\{n+1, n+2, \ldots, 2n\}$$

can divide any of the others in this set.

We can extend the pigeonhole principle by noting that if $2n+1$ pigeons fit into n pigeonholes, then at least one pigeonhole contains more than two pigeons. Also, if there are $3n+1$ pigeons in n pigeonholes, then at least one pigeonhole contains more than three pigeons. In general, we have the following.

Theorem. If m pigeons occupy n pigeonholes, then at least one pigeonhole contains

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1$$

pigeons.

proof. The largest multiple of n less than m is found by dividing $m-1$ by n and discarding the fractional part, that is,

$$\left\lfloor \frac{m-1}{n} \right\rfloor .$$

If we had exactly

$$n \left\lfloor \frac{m-1}{n} \right\rfloor \leq m-1 < m$$

pigeons, we could put $\left\lfloor \frac{m-1}{n} \right\rfloor$ in each pigeonhole. But, since we have m pigeons, at least one pigeonhole contains more than this number of pigeons.

$\square$