Computing Science 204 (B1)
Notes on Nonhomogeneous Linear Difference Equations
Wednesday, July 26

Nonhomogeneous Equations

An equation of the form

$$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_{k-1} t_{n-k+1} + a_k t_{n-k} = f(n), \quad n \geq k \qquad (*)$$

were k is a fixed positive integer and the a_i's are constant is called a *nonhomogeneous, linear, constant coefficient, k^{th}-order difference equation* or *recurrence relation.* The homogeneous equation

$$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_{k-1} t_{n-k+1} + a_k t_{n-k} = 0, \quad n \geq k \qquad (*)_0$$

is called the *associated homogeneous equation.*

As in the homogeneous case, in order to determine a solution completely, we need to specify k *initial conditions*,

$$t_0 = b_0,\ t_1 = b_1,\ \ldots,\ t_{k-1} = b_{k-1}.$$

The problem of determining a solution to the difference equation subject to these initial conitions, that is,

$$\begin{aligned} a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_{k-1} t_{n-k+1} + a_k t_{n-k} &= f(n), \quad n \geq k \\ t_0 &= b_0 \\ t_1 &= b_1 \\ &\vdots \\ t_{k-1} &= b_{k-1} \end{aligned}$$

is again called a *discrete initial value problem.*

We can prove an existence and uniqueness theorem as in the homogeneous case, provided that the function $f(n)$ is "nice" enough. In particular, we have

Theorem. (Existence and Uniqueness) If the function $f(n)$ is a solution to a homogeneous, linear, constant coefficient difference equation, then the discrete initial value problem

$$\begin{aligned} a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_{k-1} t_{n-k+1} + a_k t_{n-k} &= f(n), \quad n \geq k \\ t_0 &= b_0 \\ t_1 &= b_1 \\ &\vdots \\ t_{k-1} &= b_{k-1} \end{aligned}$$

has a unique solution for any initial values $b_0, b_1, \ldots, b_{k-1}$.

The proof is omitted, however the existence and uniqueness will again follow from the construction of the solution that follows.

The following result is used to find the most general solution to the nonhomogeneous difference equation $(*)$.

Theorem. If h_n is the general solution to the associated homogeneous equation $(*)_0$, and p_n is any *particular* solution to the nonhomogeneous equation $(*)$, then the general solution to the nonhomogeneous equation $(*)$ is

$$t_n = h_n + p_n.$$

proof. Let t_n be any solution to the equation $(*)$, and let $v_n = t_n - p_n$, then

$$\begin{aligned}
a_0 v_n + a_1 v_{n-1} + \cdots + a_k v_{n-k} &= a_0(t_n - p_n) + a_1(t_{n-1} - p_{n-1}) + \cdots + a_k(t_{n-k} - p_{n-k}) \\
&= a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} - (a_0 p_n + a_1 p_{n-1} + \cdots + a_k p_{n-k}) \\
&= f(n) - f(n) \\
&= 0,
\end{aligned}$$

and therefore v_n is a solution to the associated homogeneous equation $(*)_0$, so that

$$t_n = v_n + p_n.$$

Thus, we have shown that if t_n is any solution to the nonhomogeneous equation, then t_n can be written as the sum of a solution v_n to the homogeneous equation plus the particular solution p_n.

□

As in the homogeneous case, if the functions $f_1(n),\ f_2(n),\ \ldots,\ f_k(n)$ are linearly independent solutions to the homogeneous difference equation $(*)_0$, and p_n is any particular solution to the nonhomogeneous equation $(*)$, then the *general solution* to $(*)$ is given by

$$t_n = A_1 \cdot f_1(n) + A_2 \cdot f_2(n) + \cdots + A_k \cdot f_k(n) + p_n$$

where $A_1, A_2, \ldots, A_k$ are arbitrary constants, and *every* solution to $(*)$ can be written in this form for some values of the constants $A_1, A_2, \ldots, A_k$.

Thus, in order to solve the equation $(*)$, we need only find the general solution h_n to the homogeneous equation, and then *one* particular solution to the nonhomogeneous equation. Once this is done, to find the unique solution to the initial value problem, we use the initial conditions to determine the constants $A_1, A_2, \ldots, A_k$.

The difficult part is finding that one particular solution to the nonhomogeneous equation, and most of the methods used for that purpose in solving differential equations also work for solving difference equations, for example,

- ▶ Method of Undetermined Coefficients
- ▶ Annihilator Method
- ▶ Variation of Parameters

all work for difference equations. We will discuss only the first two methods, and both of these require that the right-hand side of the difference equation, that is, $f(n)$, be a solution to a homogeneous linear constant coefficient difference equation.

Method of Undetermined Coefficients

Suppose the right-hand side of the nonhomogeneous equation $(*)$ is

$$f(n) = Kn^d \cdot b^n,$$

where K is a constant, d is a nonnegative integer.

(i) If b is *not* a root of the characteristic equation of the associated homogeneous equation $(*)_0$, then the form of the trial solution p_n to $(*)$ should be

$$p_n = (A_0 + A_1 \cdot n + A_2 \cdot n^2 + \cdots A_{d-1} \cdot n^{d-1})b^n$$

(ii) However, if b *is* a root of the characteristic equation of the associated homogeneous equation $(*)_0$, then the form of the trial solution p_n to $(*)$ should be

$An^d \cdot b^n$ if the root b is not a repeated root of the characteristic equation.

$An^{d+1} \cdot b^n$ if the root b has multiplicity two.

$An^{d+2} \cdot b^n$ if the root b has multiplicity three, etc.

However, there may be cases where the above strategy may not work, and in this case we have to rely on the annihilator method to tell us the form of the particular solution p_n.

Example 1. Find the unique solution to the discrete initial value problem

$$\begin{aligned} a_n - 2a_{n-1} &= n2^n, \quad n \geq 1 \\ a_0 &= 1 \end{aligned}$$

solution. The homogeneous equation is $a_n - 2a_{n-1} = 0$ and the characteristic equation $\lambda - 2 = 0$ has the root $\lambda_1 = 2$, so the general solution to the homogeneous equation is $h_n = A_1 2^n$.

As a particular solution to the nonhomogeneous equation we try $p_n = An2^n + Bn^2 2^n$, so that

$$\begin{aligned} p_n - 2p_{n-1} &= An2^n + Bn^2 2^n - 2A(n-1)2^{n-1} - 2B(n-1)^2 2^{n-1} \\ &= [A - B]2^n + [2B]n2^n \end{aligned}$$

and in order for p_n to be a solution of the nonhomogeneous equation we want the following identity to hold:

$$[A - B]2^n + [2B]n2^n \equiv n2^n,$$

that is, $A - B = 0$ and $2B = 1$. Therefore, $A = B = \dfrac{1}{2}$, and a particular solution is

$$p_n = n2^{n-1} + n^2 2^{n-1}, \quad n \geq 0.$$

The general solution to the nonhomogeneous equation is

$$a_n = h_n + p_n = A_1 2^n + n2^{n-1} + n^2 2^{n-1},$$

and applying the initial condition $a_0 = 1$, we find $A_1 = 1$, so the unique solution to the initial value problem is

$$a_n = (n^2 + n + 2)2^{n-1},$$

for $n \geq 0$.

$\square$

Annihilator Method

The annihilator method allows us to solve a homogeneous linear equation (of higher order than the original) in order to determine the form of the particular solution to the nonhomogeneous equation $(*)$.

If we let D represent a difference operator which operates on a function t_n to give

$$Dt_n = t_{n+1},$$

then $(D-a)t_n = t_{n+1} - at_n$ if a is a constant. Since $(*)$ has constant coefficients, we can consider the difference equation $(*)$ to be

$$P(D)t_{n-k} = f(n)$$

where $P(D)$ is a polynomial difference operator such that

$$P(D)t_{n-k} = a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k}.$$

If the right-hand side of $(*)$ is a function $f(n)$ which is a solution to a homogeneous, linear, constant coefficient difference equation which has the form $Q(D)f_n = 0$, where $Q(D)$ is a polynomial difference operator, then operating on both sides of the equation $(*)$ with $Q(D)$ gives

$$Q(D)P(D)t_{n-k} = Q(D)f(n) = 0,$$

that is, the polynomial difference operator $Q(D)$ annihilates $f(n)$, and the solutions to this higher order homogeneous difference equation must contain a particular solution to $(*)$.

Thus, in order to find a particular solution p_n to the nonhomogeneous difference equation $(*)$, we need only find a difference operator $Q(D)$ that annihilates $f(n)$, and then solve the *homogeneous* linear equation

$$Q(D)P(D)p_{n-k} = 0.$$

In the resulting solution, any terms h_n that are solutions to the associated homogeneous equation $(*)_0$ can be omitted, since for these terms we have $P(D)h_{n-k} = 0$.

The main advantage to the annihilator method is that it allows us to find the form of the particular solution to try in the method of undetermined coefficients. If the characteristic equation of $P(D)h_{n-k} = 0$ is $p(\lambda) = 0$ and the characteristic equation of $Q(D)f(n) = 0$ is $q(\lambda) = 0$, then the characteristic equation of $Q(D)P(D)t_{n-k} = 0$ is $q(\lambda) \cdot p(\lambda) = 0$.

Example 2. Find the unique solution to the discrete initial value problem

$$\begin{aligned} a_n - 4a_{n-1} + 4a_{n-2} &= n^2 2^n, \quad n \geq 2 \\ a_0 &= 1 \\ a_1 &= 1 \end{aligned}$$

solution. The general solution to the homogeneous equation is

$$h_n = A_1 2^n + A_2 n 2^n$$

since $p(\lambda) = (\lambda - 2)^2$. The right-hand side of the nonhomogeneous equation is annihilated by a difference operator with characteristic polynomial $q(\lambda) = (\lambda - 2)^3$, and a particular solution to the nonhomogeneous equation is a solution to a homogeneous equation which has

$$q(\lambda) \cdot p(\lambda) = (\lambda - 2)^5$$

as its characteristic polynomial. Therefore, as a particular solution to the nonhomogeneous equation we should try

$$p_n = An^2 2^n + Bn^3 2^n + Cn^4 2^n$$

in the method of undetermined coefficients, and this is left as an exercise.

□