Computing Science 204 (B1)

An Application of the Fundamental Theorem of Arithmetic

Saturday, July 22

Example. *The Locker Problem*

A certain locker room contains n lockers numbered $1, 2, \ldots, n$ and they are all originally locked. An attendant performs a sequence of operations $T_1, T_2, \ldots, T_n$ whereby with the operation T_k, $1 \leq k \leq n$, the condition of being locked or unlocked is changed for all those lockers and only those lockers whose numbers are multiples of k. Show that after all the n operations have been performed, all those lockers whose numbers are perfect squares (and only those lockers) are now open or unlocked.

proof. Locker number m, for $1 \leq m \leq n$, will be unlocked after the n operations have been performed if and only if it has changed state an odd number of times, that is, if and only if the integer m has an odd number of positive divisors.

Now, if the unique prime factorization of m is given by

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1 < p_2 < \cdots < p_k$ are the prime divisors of m and $\alpha_i \geq 0$ for $i = 1, 2, \ldots, k$, then all the positive divisors d of m have the form

$$d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$

where $0 \leq \beta_i \leq \alpha_i$ for $i = 1, 2, \ldots, k$. Therefore, the number of positive divisors $\tau(m)$ of m is given by

$$\tau(m) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1),$$

and this is an odd integer if and only if each of the integers α_i is even, say, $\alpha_i = 2\gamma_i$ for $i = 1, 2, \ldots, k$.

Therefore, locker number m is unlocked after all n operations are performed if and only if

$$m = \left(p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k}\right)^2,$$

that is, if and only if m is a perfect square.