Computing Science 204 (B1)
Consecutive Fibonacci Numbers and the Euclidean Algorithm
Thursday, July 13

Recall that the *Euclidean algorithm* is used to find the greatest common divisor $\gcd(a, b)$ of two positive integers a and b, and we say that a and b are *relatively prime* if and only if $\gcd(a, b) = 1$.

Theorem.

If F_{n+1} and F_{n+2} are successive terms in the Fibonacci sequence, with $n > 1$, then F_{n+1} and F_{n+2} are relatively prime and the Euclidean algorithm takes exactly n divisions to verify that $\gcd(F_{n+1}, F_{n+2}) = 1$.

More generally, the number of divisions needed by the Euclidean algorithm to find the greatest common divisor of two positive integers does not exceed five times the number of decimal digits in the smaller of the two integers.

proof.

Applying the Euclidean algorithm to F_{n+1} and F_{n+2}, and using the defining relation for the Fibonacci numbers at each step, we have

$$\begin{aligned}
F_{n+2} &= 1 \cdot F_{n+1} + F_n \\
F_{n+1} &= 1 \cdot F_n + F_{n-1} \\
F_n &= 1 \cdot F_{n-1} + F_{n-2} \\
&\vdots \\
F_4 &= 1 \cdot F_3 + F_2 \\
F_3 &= 2 \cdot F_2
\end{aligned}$$

and the last nonzero remainder is F_2, so that the greatest common divisor $\gcd(F_{n+2}, F_{n+1}) = F_2 = 1$, and F_{n+2} and F_{n+1} are relatively prime. It is clear from the calculations above that the Euclidean algorithm takes exactly n divisions to show that $\gcd(F_{n+2}, F_{n+1}) = F_2 = 1$.

Now let a and b be positive integers with $0 < b < a$. We will apply the Euclidean algorithm to find the greatest common divisor $\gcd(a, b)$ of a and b. Let $r_0 = a$ and $r_1 = b$, then we get the following sequence of equations:

$$\begin{aligned}
r_0 &= q_1 r_1 + r_2, & 0 \le r_2 < r_1, \\
r_1 &= q_2 r_2 + r_3, & 0 \le r_3 < r_2, \\
r_2 &= q_3 r_3 + r_4, & 0 \le r_4 < r_3, \\
&\vdots \\
r_{n-3} &= q_{n-2} r_{n-2} + r_{n-1}, & 0 \le r_{n-1} < r_{n-2}, \\
r_{n-2} &= q_{n-1} r_{n-1} + r_n, & 0 \le r_n < r_{n-1}, \\
r_{n-1} &= q_n r_n.
\end{aligned}$$

Here we have used n divisions. Note that each of the quotients $q_1, q_2, \ldots, q_{n-1}$ is greater than or equal to 1, and $q_n \ge 2$, since $r_n < r_{n-1}$. Therefore,

$$\begin{aligned}
r_n &\ge 1 = F_2 \\
r_{n-1} &\ge 2r_n \ge 2F_2 = F_3 \\
r_{n-2} &\ge r_{n-1} + r_n \ge F_3 + F_2 = F_4 \\
r_{n-3} &\ge r_{n-2} + r_{n-1} \ge F_4 + F_3 = F_5 \\
&\vdots \\
r_2 &\ge r_3 + r_4 \ge F_{n-1} + F_{n-2} = F_n \\
b = r_1 &\ge r_2 + r_3 \ge F_n + F_{n-1} = F_{n+1}.
\end{aligned}$$

Thus, if there are n divisions used in the Euclidean algorithm, then we must have $b \geq F_{n+1}$. Now we show by induction that $F_{n+1} > \alpha^{n-1}$ for $n \geq 2$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

The basis step consists of verifying this inequality for $n = 2$ and $n = 3$. Since $\alpha < 2 = F_3$ then the result is true for $n = 2$. Also, since $\alpha^2 = \dfrac{3+\sqrt{5}}{2} < 3 = F_4$, the result is true for $n = 3$.

The inductive hypothesis consists of assuming that $\alpha^{k-1} < F_{k+1}$ for all integers k with $2 \leq k < n$. Since $\alpha = \dfrac{1+\sqrt{5}}{2}$ is a solution of the quadratic equation $x^2 - x - 1 = 0$, then $\alpha^2 = \alpha + 1$, so that

$$\alpha^{n-1} = \alpha^2 \cdot \alpha^{n-3} = (\alpha + 1) \cdot \alpha^{n-3} = \alpha^{n-2} + \alpha^{n-3}.$$

From the induction hypothesis, we have

$$\alpha^{n-2} < F_n \quad \text{and} \quad \alpha^{n-3} < F_{n-1},$$

and adding these inequalities, we get

$$\alpha^{n-1} = \alpha^{n-2} + \alpha^{n-3} < F_n + F_{n-1} = F_{n+1},$$

so the result is also true for $k = n$. By the principle of mathematical induction, the result is true for all $n \geq 2$.

Therefore, $b \geq F_{n+1} > \alpha^{n-1}$ for $n \geq 2$, and since $\log_{10} \alpha > \dfrac{1}{5}$, then

$$\log_{10} b > (n-1) \log_{10} \alpha > \frac{(n-1)}{5},$$

and therefore

$$n - 1 < 5 \cdot \log_{10} b.$$

Now suppose that b has k decimal digits, so that $b < 10^k$, then $\log_{10} b < k$, so that $n - 1 < 5k$. Since k is an integer, this implies that $n \leq 5k$.