Computing Science 204
The Principle of Mathematical Induction and the Well–Ordering Principle

Tuesday, July 11

Probably the most important subset of the integers $\mathbb{Z}$ is the set of *positive integers* (also called the *natural numbers* or the *counting numbers*)

$$\mathbb{N} = \{1, 2, 3, 4, \cdots\},$$

and two of the most important properties of these numbers are *The Principle of Mathematical Induction* and *The Well–Ordering Principle.*

We will show that these two properties are logically equivalent, so that we can assume one as an axiom concerning the set $\mathbb{N}$, while the other can be proved as a theorem. The simplest and most intuitive of the two is the Well–Ordering Principle, and we will assume this as an axiom for now.

- **Well–Ordering Principle:**
  Any nonempty subset of positive integers has a smallest element.
- **Principle of Mathematical Induction:**
  If P is a set of positive integers such that
  1. 1 is in P,
  2. if the integer k is in P, then the integer $k + 1$ is also in P,

  then $P = \mathbb{N}$, that is, P is the set of all positive integers.

First we prove the following lemmas:

Lemma 1. 1 is the smallest positive integer.

proof.

(i) Based on the Principle of Mathematical Induction.

Let S be the set of all positive integers greater than or equal to 1. Trivially, $1 \in S$. If the integer k is in S, then $k \geq 1$, so that

$$k + 1 > k \geq 1$$

and so the integer $k + 1$ is in S. It follows from the principle of mathematical induction that S is the set of all positive integers. Therefore, each positive integer is greater than or equal to 1.

(ii) Based on the Well–Ordering Principle.

From the Well–Ordering Principle, there is a smallest positive integer, call it s. Assume that $s < 1$, if we multiply the inequality

$$0 < s < 1$$

by s, then

$$0 < s^2 < s,$$

which implies that s is not the smallest positive integer. Because our assumption led to a contradiction, it must be false. Therefore, 1 is the smallest positive integer. □

Lemma 2. If k is an integer, there is no integer between k and $k + 1$ (exclusive).

proof.

Suppose that k is an integer and there exists an integer m such that $k < m < k + 1$, then $p = k - m$ is an integer and satisfies the inequalities $0 < p < 1$, which contradicts Lemma 1. Therefore, given an integer k, there is no integer between k and $k + 1$. □

Theorem. The principles of mathematical induction and well–ordering are equivalent.

proof.

I. Assume that the well–ordering principle holds for the positive integers. Let S be a set of positive integers with the properties

(1) 1 is in S
(2) if k is in S, then $k+1$ is also in S We have to show that S is the set of all positive integers.

Let T be the set of all positive integers that are not in S. If T is nonempty, then it follows from the well–ordering principle that T has a smallest element, say t. Since 1 is in S and 1 is the smallest positive integer, then $t > 1$. Therefore, $t-1$ is a positive integer that must be in S since $t-1 < t$. By the second property of S, we have $t-1+1 = t$ is also in S, which is a contradicton. Because our assumption that T is nonempty leads to a contradiction, it must be false. Therefore, T is empty and S is the set of all positive integers.

II. Assume that the principle of mathematical induction holds for the positive integers. Assume also that there exists a nonempty set S of positive integers that does not have a smallest element. Since 1 is the smallest positive integer, then 1 is not in S and is smaller than all of the elements of S.

Let T be the set of all positive integers that are less than all of the elements of S. We have shown that 1 is in T. Now suppose that k is in T. If $k+1$ is in S, then since there is no integer between k and $k+1$, this implies that $k+1$ is the smallest element of S, which contradicts our assumption about S. Thus, if k is in T, then $k+1$ must also be in T. It follows from the principle of mathematical induction that T is the set of all positive integers, and therefore S is empty. Therefore, if S is a nonempty set of positive integers, then S has a smallest element. □

There is a variation on the principle of mathematical induction that, in some cases, is easier to apply:

- **Second Principle of Mathematical Induction:**
  If P is a set of positive integers such that
  1. 1 is in P,
  2. if all positive integers less than k are in P, then the integer k is also in P,

  then $P = \mathbb{N}$, that is, P is the set of all positive integers.

Exercise. Prove that the second principle of mathematical induction is equivalent to both the principle of mathematical induction and the well–ordering principle.