Computing Science 204 (B1)

Notes on the Inclusion Exclusion Principle

Saturday, July 22

Suppose that we have a set S consisting of N distinct objects. Let A_1, A_2, ..., A_m be a set of properties that the objects of the set S may possess, and let $N(A_i)$ be the number of objects having property A_i. Note that an object may have several (or none) of the properties listed. Also, we let $N(A_i')$ be the number of objects that do not have the property A_i, so that

$$N = N(A_i) + N(A_i')$$

for i = 1,2, ... ,m.

Since an object can have more than one property, we have to be able to count the number of objects having both of the properties A_i and A_j, and this is denoted by $N(A_iA_j)$. The number of objects having neither of the properties A_i and A_j is denoted by $N(A_i'A_j')$. Similarly, the number of objects having property A_i but not A_j is denoted by $N(A_iA_j')$.

We shall use the following notation, which has the obvious meaning:

$$\begin{gathered} N(A_iA_j\cdots A_k), \\ N(A_i'A_j'\cdots A_k'), \\ N(A_i'A_j\cdots A_k) \\ \vdots \end{gathered}$$

The formula which gives the number of objects not having any of the m properties is called the *principle of inclusion and exclusion*, and was discoved about 100 years ago by Sylvester. It was discovered by De Moivre some years earlier in a different form.

Principle of Inclusion and Exclusion. If S is a set containing N objects, then the number of objects in S having none of the properties A_1, A_2, ..., A_m is given by

$$\begin{aligned} N(A_1'A_2'\cdots A_m') = N - \sum_i N(A_i) + \sum_{i\neq j} N(A_iA_j) - \sum_{\substack{i,j,k \\ \text{distinct}}} N(A_iA_jA_k) + \cdots \\ + (-1)^k \sum_{\substack{i_1,i_2,\ldots,i_k \\ \text{distinct}}} N(A_{i_1}A_{i_2}\cdots A_{i_k}) + \cdots + (-1)^m N(A_1A_2\cdots A_m). \end{aligned}$$

In the inclusion-exclusion formula, the first sum is over all i from $\{1, 2, \ldots, m\}$. The second sum is over all unordered pairs $\{i, j\}$ with i and j from $\{1, 2, \ldots, m\}$ and $i \neq j$. The third sum is over all unordered triples $\{i, j, k\}$ with i, j, and k from $\{1, 2, \ldots, m\}$ and i, j, k distinct.

The general term is $(-1)^k$ times a sum of terms of the form $N(A_{i_1}A_{i_2}\cdots A_{i_k})$ where the sum is over all unordered k-tuples $\{i_i, i_2, \ldots, i_k\}$ from $\{1, 2, \ldots, m\}$ with i_1, i_2, ..., i_k distinct.

proof. The left-hand side of the equation counts the number of objects in S having none of the properties. We will show that every object having none of the properties is counted exactly once in the right-hand side, and every object having at least one property is counted exactly zero times.

Suppose that an object has none of the properties A_1, A_2, ..., A_m, then it is counted in computing N, but not in $\sum N(A_i)$, $\sum N(A_iA_j)$, and so on. Therefore, it is counted exactly once in the right-hand side of the equation.

Now suppose that an object has exactly p of the properties $A_1, A_2, \ldots, A_m$, where $p \geq 1$. The object is counted

$$\begin{array}{lll} \binom{p}{0} & \text{times in} & N \\[2ex] \binom{p}{1} & \text{times in} & \sum N(A_i) \\[2ex] \binom{p}{2} & \text{times in} & \sum N(A_i A_j) \\[2ex] \binom{p}{3} & \text{times in} & \sum N(A_i A_j A_k) \\ & \vdots & \\ \binom{p}{m} & \text{times in} & N(A_1 A_2 \cdots A_m). \end{array}$$

Therefore, the (net) number of times the object is counted in the right-hand side of the equation is

$$\binom{p}{0} - \binom{p}{1} + \binom{p}{2} - \binom{p}{3} + \cdots + (-1)^m \binom{p}{m},$$

and since $p \leq m$ and $\binom{p}{k} = 0$ if $p < k$, then the total number of times the object is counted is

$$\binom{p}{0} - \binom{p}{1} + \binom{p}{2} - \binom{p}{3} + \cdots + (-1)^p \binom{p}{p} = [1 + (-1)]^p = 0.$$

Thus the object contributes a net count of 0 to the right-hand side of the equation.

□

Corollary. (Number of Onto Functions)

The number of functions from an m-element set onto an n-element set is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = n^n - \binom{n}{1}(n-1)^m + \cdots + (-1)^{n-1} \binom{n}{n-1} 1^m$$

Proof. Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, where $m \geq n$, and let S be the set of all functions $f : A \longrightarrow B$. Then the number of elements in S is n^m.

For $i = 1, 2. \ldots, n$, let a function $f \in S$ have property A_i if and only if the element b_i is *not* in the range of f. Then $N(A_i')$ counts the number of functions $f : A \longrightarrow B$ that have b_i in their range, and therefore the number of onto functions $f : A \longrightarrow B$ is given by $N(A_1' A_2' \cdots A_n')$.

Now, for each $i = 1, 2, \ldots, n$, we have $N(A_i) = (n-1)^m$. Also, for each unordered pair $\{i, j\}$ with $i \neq j$ we have $N(A_i A_j) = (n-2)^m$. Similarly, for any k-tuple $\{i_1, i_2, \ldots, i_k\}$ with $i_1, i_2, \ldots, i_k$ distinct we have $N(A_{i_1} A_{i_2} \cdots A_{i_k}) = (n-k)^m$. Therefore, from the principle of inclusion and exclusion we have

$$N(A_1' A_2' \cdots A_n') = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

□

The numbers $S(m, n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m$ are called the *Stirling numbers of the second kind.*

Instead of counting the number of onto functions directly as in the preceding corollary, we can also generate them using a recurrence relation as follows.

Theorem. Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two sets with $m \geq n$, and let $a(m,n)$ count the number of *onto* functions $f : A \longrightarrow B$ from A to B. Then $a(m,n)$ satisfies the discrete initial value problem

$$\begin{aligned} a(m,n) &= n^m - \sum_{k=1}^{n-1} \binom{n}{k} a(m,k), \qquad m \geq n > 1 \\ a(m,1) &= 1 \end{aligned}$$

Proof. If $n = 1$, then $B = \{b\}$, that is, B is a singleton, and every onto function $f : A \longrightarrow B$ satisfies $f(a) = b$ for all $a \in A$. Therefore, $a(m,1) = 1$.

Now, if $m \geq n > 1$, then the total number of functions $f : A \longrightarrow B$ is n^m, and if $1 \leq k \leq n-1$, then there are exactly $\binom{n}{k} a(m,k)$ onto functions $g : A \longrightarrow B$ with domain A and range a subset of B of size k.

Also, any function $h : A \longrightarrow B$ that is not onto B is found among these functions g. Therefore,

$$a(m,n) = n^m - \sum_{k=1}^{n-1} \binom{n}{k} a(m,k)$$

for $m \geq n > 1$.

$\square$