**Computing Science 204 (B1)**
**Notes on Homogeneous Linear Difference Equations**
Tuesday, July 25

### Homogeneous Equations

An equation of the form

$$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_{k-1} t_{n-k+1} + a_k t_{n-k} = 0, \quad n \geq k \qquad (*)$$

were $k$ is a fixed positive integer and the $a_i$'s are constant is called a *homogeneous, linear, constant coefficient,* $k^{\text{th}}$*-order difference equation* or *recurrence relation.*

In order to determine a solution completely, we need to specify $k$ *initial conditions*,

$$t_0 = b_0,\ t_1 = b_1,\ \ldots,\ t_{k-1} = b_{k-1}.$$

The problem of determining a solution to the difference equation subject to these initial conitions, that is,

$$\begin{aligned} a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_{k-1} t_{n-k+1} + a_k t_{n-k} &= 0, \quad n \geq k \\ t_0 &= b_0 \\ t_1 &= b_1 \\ &\vdots \\ t_{k-1} &= b_{k-1} \end{aligned}$$

is called a *discrete initial value problem.*

**Theorem.** (Existence and Uniqueness) The discrete initial value problem

$$\begin{aligned} a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \cdots + a_{k-1} t_{n-k+1} + a_k t_{n-k} &= 0, \quad n \geq k \\ t_0 &= b_0 \\ t_1 &= b_1 \\ &\vdots \\ t_{k-1} &= b_{k-1} \end{aligned}$$

has a unique solution for any initial values $b_0, b_1, \ldots, b_{k-1}$.

The proof is omitted, however the existence and uniqueness will follow from the construction of the solution that follows.

The following result is used to find the most general solution to the difference equation $(*)$.

**Theorem.** (Principle of Superposition) If $f_n$ and $g_n$ are solutions to the difference equation $(*)$, then any linear combination of these solutions,

$$t_n = A \cdot f_n + B \cdot g_n$$

where $A$ and $B$ are constants, is also a solution to the equation $(*)$.
**proof.** For all $n \geq k$, we have

$$\begin{aligned} a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} &= a_0(A \cdot f_n + B \cdot g_n) + a_1(A \cdot f_{n-1} + B \cdot g_{n-1}) + \cdots + a_k(A \cdot f_{n-k} + B \cdot g_{n-k}) \\ &= A \cdot (a_0 f_n + a_1 f_{n-1} + \cdots + a_k f_{n-k}) + B \cdot (a_0 g_n + a_1 g_{n-1} + \cdots + a_k g_{n-k}) \\ &= A \cdot 0 + B \cdot 0 = 0. \end{aligned}$$

$\square$

If $f_1(n),\ f_2(n),\ \ldots,\ f_k(n)$ are linearly independent solutions to the difference equation $(*)$, that is, none of them is a linear combination of the others, then the *general solution* to $(*)$ is defined to be

$$t_n = A_1 \cdot f_1(n) + A_2 \cdot f_2(n) + \cdots + A_k \cdot f_k(n)$$

where $A_1, A_2, \ldots, A_k$ are arbitrary constants, and *every* solution to $(*)$ can be written in this form for some values of the constants $A_1, A_2, \ldots, A_k$.

In order to solve the equation $(*)$, we assume a solution of the form

$$t_n = \lambda^n,$$

and try to determine the value of the real or complex number $\lambda$ which forces this to be a solution to $(*)$. Substituting this expression for $t_n$ into the difference equation, we get

$$a_0\lambda^n + a_\lambda^{n-1} + \cdots + a_{k-1}\lambda^{n-k+1} + a_k\lambda^{n-k} = 0,$$

assuming that $\lambda \neq 0$, we get the *characteristic equation* or *auxiliary equation* associated with the difference equation $(*)$

$$p(\lambda) = a_0\lambda^k + a_1\lambda^{k-1} + \cdots + a_{k-1}\lambda + a_k = 0.$$

The polynomial of degree $k$ is called the *characteristic polynomial* of $(*)$, and from the Fundamental Theorem of Algebra, it has exactly $k$ roots, $\lambda_1, \lambda_2, \ldots, \lambda_k$, which may be real or complex numbers. The characteristic polynomial can be factored as

$$p(\lambda) = a_0(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_k)$$

and since $p(\lambda_i) = 0$ for $i = 1, 2, \ldots, k$, all of the functions

$$\lambda_1^n,\ \lambda_2^n,\ \ldots,\ \lambda_k^n$$

are solutions to the difference equation $(*)$.

Therefore, if we can find $k$ linearly independent solutions to $(*)$, say, $f_1(n), f_2(n), \ldots, f_k(n)$, then we can write down the general solution

$$t_n = A_1 f_1(n) + A_2 f_2(n) + \cdots + A_k f_k(n),$$

and then use the initial conditions to determine the constants to find the unique solution to the initial value problem.

**The Case of Distinct Real Roots**

It is not difficult to show that if the roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, then the functions

$$f_1(n) = \lambda_1^n,\ f_2(n) = \lambda_2^n,\ \ldots,\ f_k(n) = \lambda_k^n$$

are linearly independent, so that the general solution to $(*)$ is given by

$$t_n = A_1\lambda_1^n + A_2\lambda_2^n + \cdots A_k\lambda_k^n$$

where the constants $A_1, A_2, \ldots, A_k$ are found from the initial conditions $t_0 = b_0,\ t_1 = b_1,\ \ldots,\ t_{k-1} = b_{k-1}$.

**Example 1.** Solve the discrete initial value problem

$$\begin{aligned} a_n - 3a_{n-1} + 2a_{n-2} &= 0, \quad n \geq 2 \\ a_0 &= 0 \\ a_1 &= 1 \end{aligned}$$

**solution.** Assuming a solution of the form $a_n = \lambda^n$ we get the characteristic equation

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0,$$

with distinct real roots $\lambda_1 = 1$ and $\lambda_2 = 2$, and since the functions $f_1(n) = 1^n = 1$ and $f_2(n) = 2^n$ are linearly independent, then the general solution is

$$a_n = A \cdot \lambda_1^n + B \cdot \lambda_2^n = A + B \cdot 2^n.$$

Applying the initial conditions, we get

$$\begin{aligned} 0 = a_0 &= A + B \\ 1 = a_1 &= A + 2B \end{aligned}$$

so that $B = -A = 1$, and the solution to the initial value problem is

$$a_n = 2^n - 1$$

for $n \geq 0$.

$\square$

**The Case of Repeated Real Roots**

If the distinct real roots of the characteristic equation $p(\lambda) = 0$ of $(*)$ are $\lambda_1, \lambda_2, \ldots, \lambda_m$, with multiplicities $d_1, d_2, \ldots, d_m$, respectively, where $d_1 + d_2 + \cdots + d_m = k$, then it is not difficult to check that the functions

$$\lambda_i^n,\ n \cdot \lambda_i^n,\ n^2 \cdot \lambda_i^n,\ \ldots,\ n^{d_i - 1} \cdot \lambda_i^n, \qquad 1 \le i \le m,$$

are linearly independent, and the contribution of the root $\lambda_i$ to the general solution of $(*)$ is

$$A_{i,1}\lambda_i^n + A_{i,2}n\lambda_i^n + \cdots + A_{i,d_i}n^{d_i-1}\lambda_i^n$$

where the $k$ constants $A_{i,1}, A_{i,2}, \ldots, A_{i,d_i}$, for $1 \le i \le m$ are to be determined from the $k$ initial conditions.

**Example 2.** Solve the discrete initial value problem

$$\begin{aligned} a_n - 4a_{n-1} + 4a_{n-2} &= 0, \quad n \ge 2 \\ a_0 &= 0 \\ a_1 &= 1 \end{aligned}$$

**solution.** Assuming a solution of the form $a_n = \lambda^n$, the characteristic equation is

$$p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0,$$

with equal roots $\lambda_1 = \lambda_2 = 2$, and since the functions $f_1(n) = 2^n$ and $f_2(n) = n2^n$ are linearly independent solutions to the difference equation, then the general solution to the difference equation is

$$a_n = A \cdot 2^n + B \cdot n2^n.$$

Applying the initial conditions, we find

$$\begin{aligned} 0 &= a_0 = A \\ 1 &= a_1 = 2A + 2B \end{aligned}$$

so that $A = 0$ and $B = \dfrac{1}{2}$. Therefore, the solution to the initial value problem is

$$a_n = n \cdot 2^{n-1}$$

for $n \ge 0$.

$\square$

**The Case of Complex Roots**

We will not consider the case of complex roots here, although one such illustrative example appears in the notes section on my web page.