Computing Science 204 (B1)

The Harmonic Numbers

Tuesday, July 18

The *Harmonic numbers* $H_1, H_2, H_3, \ldots, H_n, \ldots$ are defined by

$$\begin{aligned} H_1 &= 1 \\ H_2 &= 1 + \frac{1}{2} \\ H_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ &\vdots \\ H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &\vdots \end{aligned}$$

we will prove the following properties of the Harmonic numbers:

(a) $H_{2^n} < 1 + n$ for all $n \geq 1$.

(b) $H_{2^n} \geq 1 + \dfrac{n}{2}$ whenever n is a nonnegative integer.

(c) The *Harmonic Series* $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, that is, $\lim_{n\to\infty} H_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = +\infty$.

(d) $\sum_{k=1}^{n} H_k = (n+1)H_n - n$ for all $n \geq 1$.

(e) If $a_n = H_n - \log(n+1)$ for $n \geq 1$, then $\lim_{n\to\infty} a_n = \gamma$ exists and $0 < \gamma < 1$.

proof.

(a) For $n = 1$, the left hand side of the inequality is $H_2 = 1 + 1/2 = 3/2$, while the right side is $1 + 1 = 2$, and the result is true for $n = 1$. Assuming that the result is true for some $n \geq 1$, we have

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \underbrace{\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n}}_{2^n \text{ terms, each } < \frac{1}{2^n}} \\ &< n + 1 + 1 \\ &= n + 2 \end{aligned}$$

so the result is true for $n + 1$. By the principle of mathematical induction, it is true for all $n \geq 1$.

(b) For $n = 0$, the left hand side of the inequality is $H_1 = 1$, while the right side is $1 + 0/2 = 1$, so the result holds for $n = 0$. Assuming the result is true for some $n \geq 0$, we have

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \underbrace{\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n}}_{2^n \text{ terms, each } \geq \frac{1}{2^{n+1}}} \\ &> 1 + \frac{n}{2} + 2^n \cdot \frac{1}{2^{n+1}} \\ &= 1 + \frac{n}{2} + \frac{1}{2} \\ &= 1 + \frac{n+1}{2} \end{aligned}$$

and the result is true for $n+1$. By the principle of mathematical induction, the result is true for all $n \geq 0$.

(c) From part (b) we have

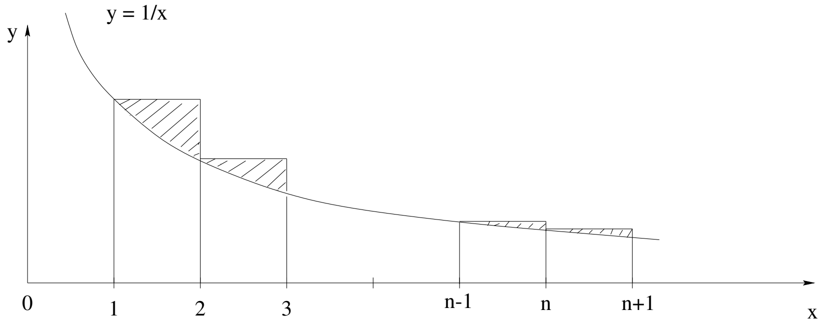
$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} &= \lim_{n\to\infty} H_n \\ &\geq \lim_{n\to\infty}\left(1+\frac{n}{2}\right) \\ &= +\infty\end{aligned}$$

(d) For $n = 1$, the left side is $H_1 = 1$, while the right side is $(1+1)H_1 - 1 = 2\cdot 1 - 1 = 1$, and the result is true for $n = 1$. Assuming the result is true for some integer $n \geq 1$, we have

$$\begin{aligned}\sum_{k=1}^{n+1} H_k = \sum_{k=1}^{n} H_k + H_{n+1} &= (n+1)H_n - n + H_{n+1} \\ &= (n+1)H_{n+1} - 1 - n + H_{n+1} = (n+2)H_{n+1} - (n+1)\end{aligned}$$

and the result is true for $n+1$. By the principle of mathematical induction, the result is true for all $n \geq 1$.

(e) Examining the area under the graph of $y = \dfrac{1}{x}$, $1 \leq x \leq n+1$,



we see that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx = \log(n+1),$$

that is, $a_n = H_n - \log(n+1) > 0$ for all $n \geq 1$.

Now, since $\log(1+x) < x$ for all $x > -1$, we have

$$\begin{aligned}a_{n+1} - a_n &= \frac{1}{n+1} - \log(n+2) + \log(n+1) \\ &= \frac{1}{n+1} - \log\left(\frac{n+2}{n+1}\right) \\ &= \frac{1}{n+1} - \log\left(1 + \frac{1}{n+1}\right) \\ &> 0,\end{aligned}$$

that is,

$$a_{n+1} - a_n = \frac{1}{n+1} - \log\left(1 + \frac{1}{n+1}\right) > 0$$

and the sequence $\{a_n\}_{n\geq 1}$ is strictly increasing.

Also, since the area of the k^{th} shaded "triangle", less than

$$\frac{1}{k} - \frac{1}{k+1},$$

for $k \geq 1$, we have

$$a_n < \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right),$$

that is $0 < a_n < 1 - \dfrac{1}{n+1}$.

Therefore, the sequence $\{a_n\}_{n \geq 1}$ is strictly increasing and bounded above, and hence converges, so the limit

$$\gamma = \lim_{n \to \infty} a_n$$

exists and $0 < \gamma < 1$.

$\square$

Note: γ is called the *Euler-Mascheroni Constant* and

$$\gamma = 0.5772156649 \cdots$$

It is not known if γ is rational or irrational.