Computing Science 204
Notes on the Fibonacci Sequence
Wednesday, July 12

The **Fibonacci sequence** $\{F_n\}_{n\geq 0}$ is the unique solution to the discrete initial value problem

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n, \qquad n \geq 0 \\ F_0 &= 0 \\ F_1 &= 1. \end{aligned}$$

The first few terms of the sequence are:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

We proved **Cassini's Identity**:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

for all integers $n \geq 1$, by evaluating the determinant of the matrix

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, in two different ways.

Here we will use Cassini's identity to show that the sequence

$$\gamma_n = \left\{ \frac{F_n}{F_{n+1}} \right\}_{n\geq 1}$$

converges and that $\lim_{n\to\infty} \gamma_n = \dfrac{\sqrt{5}-1}{2}$.

After examining the terms

$$\frac{F_{2n}}{F_{2n+1}} - \frac{F_{2n-2}}{F_{2n-1}},$$

it appears that they are all positive, and that $F_{2n} \cdot F_{2n-1} - F_{2n+1} \cdot F_{2n-2} = 1$ for all $n \geq 1$. For example, if $n = 1$, we have

$$F_2 \cdot F_1 - F_3 \cdot F_0 = 1 \cdot 1 - 2 \cdot 0 = 1.$$

Lemma 1. For each $n \geq 1$,

$$F_{2n} \cdot F_{2n-1} - F_{2n+1} \cdot F_{2n-2} = 1, \qquad (*)$$

so that

$$0 < \gamma_{2n-2} = \frac{F_{2n-2}}{F_{2n-1}} < \frac{F_{2n}}{F_{2n+1}} = \gamma_{2n} < 1$$

for all $n \geq 2$. Therefore the sequence $\{\gamma_{2n}\}_{n\geq 1}$ is strictly increasing and bounded above, hence converges.

proof. If $n \geq 1$, then

$$\begin{aligned} F_{2n} \cdot F_{2n-1} - F_{2n+1} \cdot F_{2n-2} &= \left[F_{2n-1} + F_{2n-2}\right] \cdot F_{2n-1} - \left[F_{2n} + F_{2n-1}\right] \cdot F_{2n-2} \\ &= F_{2n-1}^2 + F_{2n-2} \cdot F_{2n-1} - F_{2n} \cdot F_{2n-2} - F_{2n-1} \cdot F_{2n-2} \\ &= F_{2n-1}^2 - F_{2n} \cdot F_{2n-2} \\ &= -(-1)^{2n-1} \\ &= 1 \end{aligned}$$

by Cassini's identity. Therefore, $(*)$ is true for all $n \geq 1$.

$\square$

After examining the terms

$$\frac{F_{2n-1}}{F_{2n}} - \frac{F_{2n+1}}{F_{2n+2}},$$

it appears that they are all positive, and that $F_{2n+2} \cdot F_{2n-1} - F_{2n} \cdot F_{2n+1} = 1$ for all $n \geq 1$. For example, if $n = 1$, we have

$$F_4 \cdot F_1 - F_2 \cdot F_3 = 3 \cdot 1 - 1 \cdot 2 = 1.$$

Lemma 2. For each $n \geq 1$,

$$F_{2n+2} \cdot F_{2n-1} - F_{2n} \cdot F_{2n+1} = 1, \tag{$**$}$$

so that

$$0 < \gamma_{2n+1} = \frac{F_{2n+1}}{F_{2n+2}} < \frac{F_{2n-1}}{F_{2n}} = \gamma_{2n-1} < 1$$

for all $n \geq 2$. Therefore the sequence $\{\gamma_{2n-1}\}_{n \geq 1}$ is strictly decreasing and bounded below, hence converges.

proof. If $n = 1$, then

$$\begin{aligned}
F_{2n+2} \cdot F_{2n-1} - F_{2n} \cdot F_{2n+1} &= \left[F_{2n+1} + F_{2n}\right] \cdot F_{2n-1} - F_{2n} \cdot \left[F_{2n} + F_{2n-1}\right] \\
&= F_{2n+1} \cdot F_{2n-1} + F_{2n} \cdot F_{2n-1} - F_{2n}^2 - F_{2n}F_{2n-1} \\
&= F_{2n+1} \cdot F_{2n-1} - F_{2n}^2 \\
&= (-1)^{2n} \\
&= 1
\end{aligned}$$

by Cassini's identity. Therefore, $(**)$ is true for all $n \geq 1$.

$\square$

Now let

$$u = \lim_{n\to\infty} \gamma_{2n} \qquad \text{and} \qquad v = \lim_{n\to\infty} \gamma_{2n+1},$$

then from Cassini's identity, we have

$$F_{2n+1} \cdot F_{2n-1} - F_{2n}^2 = (-1)^{2n} = 1,$$

so that

$$\gamma_{2n-1} - \gamma_{2n} = \frac{F_{2n-1}}{F_{2n}} - \frac{F_{2n}}{F_{2n+1}} = \frac{(-1)^{2n}}{F_{2n} \cdot F_{2n+1}},$$

and letting $n \to \infty$, we have $v - u = 0$, that is, $u = v$.

Now, given $\epsilon > 0$, there exists an integer N_1 such that $|u - \gamma_{2n}| < \epsilon$ for all positive integers n such that $2n > N_1$, and there exists an integer N_2 such that $|v - \gamma_{2n+1}| < \epsilon$ for all positive integers n such that $2n+1 > N_2$. Let $N = \max\{N_1, N_2\}$, since $u = v$, we have $|u - \gamma_n| < \epsilon$ for all $n > N$.

So we have shown that given any $\epsilon > 0$, there exists an integer N such that $|u - \gamma_n| < \epsilon$ whenever $n > N$, but this is exactly what we mean when we say $\lim_{n\to\infty} \gamma_n = u$.

Now we know that the limit $u = \lim_{n\to\infty} \gamma_n$ exists, but we still don't know its value. If we look at the difference equation for the Fibonacci numbers, $F_{n+2} = F_{n+1} + F_n$, and divide both sides by F_{n+1}, then we have

$$\frac{F_{n+2}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}}$$

for all $n \geq 1$, that is,

$$\frac{1}{\dfrac{F_{n+1}}{F_{n+2}}} = 1 + \frac{F_n}{F_{n+1}}$$

for all $n \geq 1$, and letting $n \to \infty$, we get

$$\frac{1}{u} = 1 + u.$$

Therefore, u satisfies the quadratic equation $u^2 + u - 1 = 0$, with distinct real roots

$$\lambda_1 = \frac{-1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{-1-\sqrt{5}}{2}.$$

Since each term in the sequence satisfies $0 < \gamma_n < 1$, then the limit is the positive root

$$u = \lim_{n\to\infty} \gamma_n = \frac{\sqrt{5}-1}{2}.$$

It follows from this that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

the **Golden Ratio.**