The Fibonacci Numbers and Binomial Coefficients

Wednesday, July 12, 2000

Recall that the **Fibonacci sequence** $\{F_n\}_{n\geq 0}$ is the unique solution to the discrete initial value problem

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n, \qquad n \geq 0 \\ F_0 &= 0 \\ F_1 &= 1. \end{aligned}$$

Define $F(x) = \sum_{n=0}^{\infty} F_n x^n$, $-\beta < x < \beta$, where $\beta = \frac{\sqrt{5}-1}{2}$, it can be shown that the series converges for all such real numbers x. The function F is called the **generating function** of the Fibonacci sequence $\{F_n\}_{n\geq 0}$.

We will show first that $F(x) = \dfrac{x}{1-x-x^2}$, for $-\beta < x < \beta$.

For each nonnegative integer k, we have $F_{k+2} = F_{k+1} + F_k$, multiplying this equation by x^{k+2}, we get

$$F_{k+2}x^{k+2} = F_{k+1}x^{k+2} + F_k x^{k+2},$$

and summing over k, we get

$$\sum_{k=0}^{\infty} F_{k+2}x^{k+2} = x \cdot \sum_{k=0}^{\infty} F_{k+1}x^{k+1} + x^2 \cdot \sum_{k=0}^{\infty} F_k x^k.$$

Therefore,

$$F(x) - F_0 - F_1 x = x(F(x) - F_0) + x^2 F(x),$$

so that

$$(1 - x - x^2)F(x) = F_0 + (F_1 - F_0)x = x,$$

since $F_0 = 0$ and $F_1 = 1$. Solving for $F(x)$,

$$F(x) = \frac{x}{1-x-x^2}$$

for $-\beta < x < \beta$.

Given that the geometric series

$$\sum_{n=0}^{\infty} a^n = \lim_{N\to\infty} \frac{1-a^N}{1-a} = \frac{1}{1-a}$$

converges for $-1 < a < 1$, we will use this and the binomial theorem to show that

$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

for all $n \geq 0$.

Writing

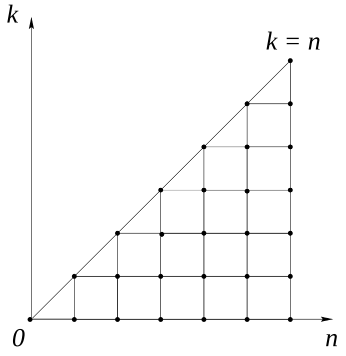
$$\frac{1}{1-x-x^2} = \frac{1}{1-x(1+x)},$$

this is the sum of the geometric series $\sum_{n=0}^{\infty} x^n(1+x)^n$, and converges for $-\beta < x < \beta$, since then $|x(1+x)| < 1$.

Therefore, using the binomial theorem to expand the powers $(1+x)^n$, we have

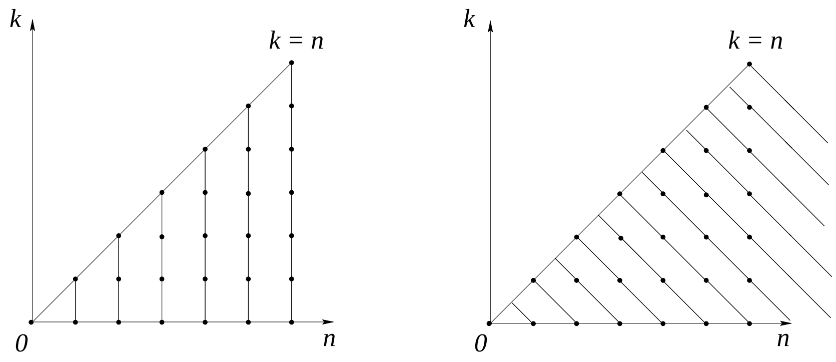
$$F(x) = \frac{x}{1-x-x^2} = x \cdot \sum_{n=0}^{\infty} x^n (1+x)^n = x \cdot \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k} x^k = x \cdot \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n+k}. \qquad (*)$$

In the last sum, we are summing over all lattice points (points with integer coordinates) in the n,k–plane which lie on or below the line $k=n$. In the inner sum, k goes from 0 to the line $k=n$, and in the outer sum, n goes from 0 to ∞.



If, instead of summing vertically over k and then summing over n, as shown on the left below, we sum diagonally, over the lines $n+k=p$, as p goes from 0 to ∞, then we can write this as a sum of the form

$$F(x) = x \cdot \sum_{p=0}^{\infty} a_p x^p.$$



Now we can deduce that for each nonnegative integer p, the coefficient of x^{p+1} in this latter sum is given by $a_p = F_{p+1}$. If we don't worry about convergence, this can be done by collecting coefficients that multiply like powers of x in $(*)$:

$$F(x) = x \cdot \sum_{p=0}^{\infty} \left\{ \sum_{n+k=p} \binom{n}{k} \right\} x^p = x \cdot \sum_{p=0}^{\infty} \left\{ \sum_{\ell=0}^{p} \binom{p-\ell}{\ell} \right\} x^p$$

where the last equality holds since for each $p \geq 0$, we are summing over only those n's and k's such that $n+k=p$. Therefore, for all $p \geq 0$, we have $F_{p+1} = \sum_{\ell=0}^{p} \binom{p-\ell}{\ell}$.