Computing Science 204 (B1)

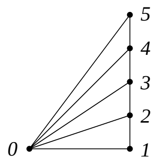
Counting Spanning Trees: An Example

Thursday, August 10

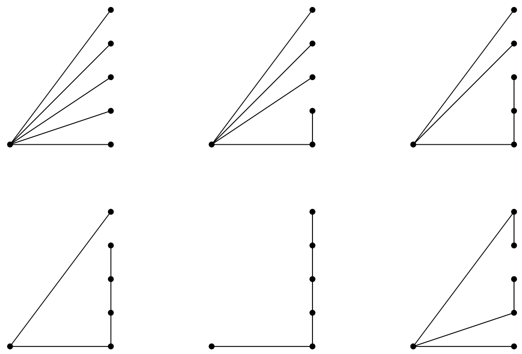
Example. *Fan Graphs and Their Spannning Trees*

A *fan* of order n is a graph on $n+1$ vertices, labeled $\{0, 1, 2 \ldots, n\}$, with $2n-1$ edges defined as follows: Vertex 0 is connected by an edge to each of the other n vertices, and vertex k is connected by an edge to vertex $k+1$, for $1 \leq k < n$.

For example, the fan of order 5, which has six vertices and nine edges, is shown below.



A *spanning tree* in a graph G is a subgraph T containing all of the vertices of G which is connected and contains no cycles. For example, the figure below shows a few of the spanning trees for the fan shown above.



Let a_n be the number of spanning trees for a fan of order n.

(a) Calculate a_1, a_2, and a_3, and show all the spanning trees in each case.

(b) By observing how the topmost vertex (vertex n) is connected to the rest of the spanning tree, show that a_n satisfies the full-history recurrence relation

$$a_n = 1 + a_{n-1} + \sum_{k=1}^{n-1} a_k$$

for $n \geq 1$, where $a_0 = 0$ and $a_1 = 1$.

(c) Find the generating function

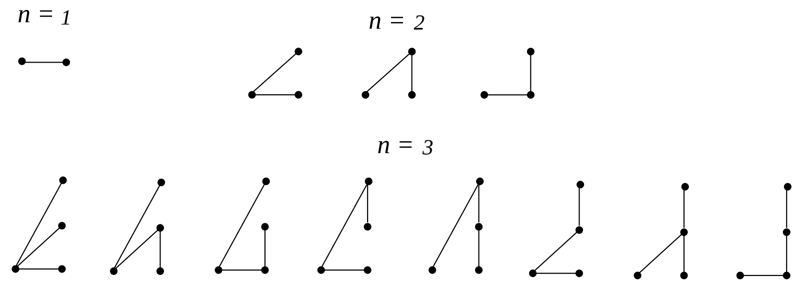
$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

for the sequence $\{a_n\}_{n \geq 0}$.

(d) By comparing the generating function found above to the generating function of a well-known sequence $\{b_n\}_{n \geq 0}$, find an explicit formula for a_n, $n \geq 0$.

Solution. Let a_n be the number of spanning trees for a fan of order n.

(a) The spanning trees for $n = 1$, $n = 2$, $n = 3$ are shown below

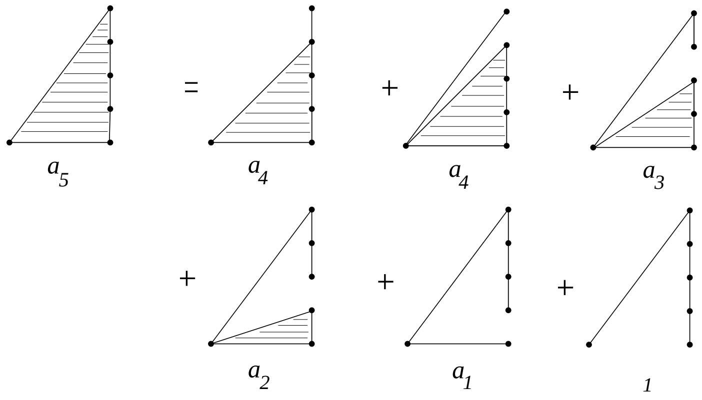


and is clear that $a_1 = 1$, $a_2 = 3$, and $a_3 = 8$.

(b) Given a fan of order n, if the topmost vertex is not connected to vertex 0, then it must be connected to vertex $n-1$, since the graph is connected. In this case, any of the a_{n-1} spanning trees for the remaining fan (on the vertices 0 through $n-1$) will complete a spanning tree for the entire graph.

If the topmost vertex is connected to vertex 0, then there exists an integer k, with $k \leq n$, such that the vertices $n, n-1, \ldots, k$ are connected directly to vertex 0, but the edge between k and $k-1$ is not in the subtree. In this case, there cannot be any edges between vertex 0 and the vertices $\{n-1, \ldots, k\}$, or there would be a cycle.

If $k = 1$, the subtree is uniquely determined, and if $k > 1$, then any of the a_{k-1} ways to produce a spanning tree on $\{0, 1, \ldots, k-1\}$ will give a spanning tree for the entire graph. For example, when $n = 5$, we have



Therefore, in general,

$$a_n = a_{n-1} + a_{n-1} + a_{n-2} + a_{n-3} + \cdots + a_1 + 1,$$

that is,

$$a_n = a_{n-1} + \sum_{k=1}^{n-1} a_k + 1,$$

and if we define $a_0 = 0$, then

$$a_n = a_{n-1} + \sum_{k=0}^{n-1} a_k + 1 \qquad (*)$$

for all $n \geq 1$.

(c) Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n\}_{n \geq 0}$. Multiplying $(*)$ by x^n we get

$$a_n x^n = a_{n-1} x^n + \left(\sum_{k=0}^{n-1} a_k \right) x^n + x^n,$$

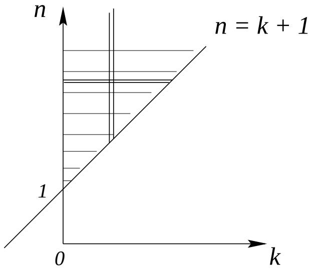
and summing from $n = 1$ to ∞, we have

$$\sum_{n=1}^{\infty} a_n x^n = x \cdot \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} a_k \right) x^n + \sum_{n=1}^{\infty} x^n,$$

that is,

$$F(x) - a_0 = x \cdot F(x) + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} a_k \right) x^n + \frac{x}{1-x}. \qquad (**)$$

In the sum $\sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} a_k \right) x^n$ we are summing over the lattice points in the (k, n)-plane in the infinite triangular region shown below



and if we change the order of summation (summing over the same lattice points), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} a_k \right) x^n &= \sum_{k=0}^{\infty} a_k \left(\sum_{n=k+1}^{\infty} x^n \right) \\ &= \sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{1-x} \\ &= \frac{x}{1-x} \cdot \sum_{k=0}^{\infty} a_k x^k \\ &= \frac{x}{1-x} \cdot F(x). \end{aligned}$$

Therefore, from $(**)$, we have

$$F(x) - a_0 = x \cdot F(x) + \frac{x}{1-x} \cdot F(x) + \frac{x}{1-x},$$

that is,

$$F(x) \left[1 - x - \frac{x}{1-x} \right] = \frac{x}{1-x},$$

that is,

$$F(x) = \frac{x}{(1-x)^2 - x} = \frac{x}{1 - 3x + x^2}.$$

(d) Note that $F(x) = \dfrac{x}{1 - 3x + x^2}$ is the generating sequence for the sequence $\{b_n\}_{n \geq 0}$, where

$$b_n = F_{2n}$$

for $n \geq 0$, that is, the sequence consisting of the Fibonacci numbers with even index.

Note also that this is not the only way to solve this problem. We can find another recurrence relation satisfied by the sequence $\{a_n\}_{\geq 0}$ as follows, we write down the recurrence relation for n and also for $n+1$, and subtract the first from the second to get

$$a_{n+1} - a_n = a_n - a_{n-1} + a_n,$$

that is,

$$a_{n+1} = 3a_n - a_{n-1}$$

together with the initial conditions $a_0 = 0$ and $a_1 = 1$, and now we recognize this as the initial value problem satisfied by the sequence $\{F_{2n}\}_{n \geq 0}$, and use the uniqueness theorem to see that $a_n = F_{2n}$ for all $n \geq 0$.