Computing Science 204 (B1)

The Method of False Position and Fibonacci Numbers
Thursday, July 13

The *Method of False Position* for solving the equation

$$f(x) = 0$$

starts with two approximations x_0 and x_1 and finds successively better approximations $x_2, x_3, x_4, \ldots$ from the formula

$$x_{n+1} = \frac{x_{n-1}f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

If we start with the function

$$f(x) = x^2$$

and the initial values

$$x_0 = 1 \qquad \text{and} \qquad x_1 = \tfrac{1}{2},$$

then

$$x_n = \frac{1}{F_{n+2}}$$

for $n \geq 0$.

proof. If $f(x) = x^2$, the recursion relation for the approximations becomes

$$x_{n+1} = \frac{x_{n-1}x_n^2 - x_n x_{n-1}^2}{x_n^2 - x_{n-1}^2} = \frac{x_{n-1}x_n}{x_n + x_{n-1}},$$

and therefore,

$$\frac{1}{x_{n+1}} = \frac{1}{x_n} + \frac{1}{x_{n-1}}.$$

Let $b_n = \dfrac{1}{x_n}$ for $n \geq 0$, then the sequence $\{b_n\}_{n\geq 0}$ satisfies the following discrete initial value problem

$$\begin{aligned} b_{n+1} &= b_n + b_{n-1} \\ b_0 &= 1 \\ b_1 &= 2 \end{aligned}$$

whose unique solution is $b_n = F_{n+2}$, and therefore

$$x_n = \frac{1}{F_{n+2}}$$

for $n \geq 0$, where F_{n+2} is the $(n+2)^{\text{nd}}$ term in the Fibonacci sequence.

$\square$