Computing Science 204 (B1)
The Last Two Examples
Wednesday, August 16

Example 1. Show that for all integers $n \geq 1$, the integer

$$\left\lceil \left(\sqrt{3}+1\right)^{2n} \right\rceil$$

is divisible by 2^{n+1}.

Solution. We show first that

$$\left\lceil \left(\sqrt{3}+1\right)^{2n} \right\rceil = \left(1+\sqrt{3}\right)^{2n} + \left(1-\sqrt{3}\right)^{2n}$$

for all $n \geq 1$. First note that for every positive integer n we have

$$\begin{aligned}
\left(1+\sqrt{3}\right)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} 3^{\frac{k}{2}} \\
&= \sum_{\substack{k=0 \\ k \text{ even}}}^{2n} \binom{2n}{k} 3^{\frac{k}{2}} + \sum_{\substack{k=0 \\ k \text{ odd}}}^{2n} \binom{2n}{k} 3^{\frac{k}{2}} \\
&= \sum_{\substack{k=0 \\ k \text{ even}}}^{2n} \binom{2n}{k} 3^{\frac{k}{2}} + \sqrt{3} \sum_{\substack{k=0 \\ k \text{ odd}}}^{2n} \binom{2n}{k} 3^{\frac{k-1}{2}} \\
&= A_n + \sqrt{3} B_n
\end{aligned}$$

where A_n and B_n are positive integers.

Similarly, for any positive integer n we have

$$\begin{aligned}
\left(1-\sqrt{3}\right)^{2n} &= \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} 3^{\frac{k}{2}} \\
&= \sum_{\substack{k=0 \\ k \text{ even}}}^{2n} \binom{2n}{k} 3^{\frac{k}{2}} - \sum_{\substack{k=0 \\ k \text{ odd}}}^{2n} \binom{2n}{k} 3^{\frac{k}{2}} \\
&= \sum_{\substack{k=0 \\ k \text{ even}}}^{2n} \binom{2n}{k} 3^{\frac{k}{2}} - \sqrt{3} \sum_{\substack{k=0 \\ k \text{ odd}}}^{2n} \binom{2n}{k} 3^{\frac{k-1}{2}} \\
&= A_n - \sqrt{3} B_n .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(1+\sqrt{3}\right)^{2n} &= A_n + \sqrt{3} B_n \\
\left(1-\sqrt{3}\right)^{2n} &= A_n - \sqrt{3} B_n
\end{aligned}$$

so that

$$\left(1+\sqrt{3}\right)^{2n} + \left(1-\sqrt{3}\right)^{2n} = 2A_n$$

is a positive integer for all $n \geq 1$. Now, since $|1-\sqrt{3}| < 1$, then $0 < \left(1-\sqrt{3}\right)^{2n} < 1$ for all $n \geq 1$, so that

$$2A_n - 1 < \left(1+\sqrt{3}\right)^{2n} \leq 2A_n \qquad \text{and} \qquad 2A_n = \left\lceil \left(\sqrt{3}+1\right)^{2n} \right\rceil$$

for all $n \geq 1$. We will prove by induction that both A_n and B_n are divisible by 2^n for all $n \geq 1$.

For $n = 1$ we have $\left(1+\sqrt{3}\right)^2 = 4 + 2\sqrt{3}$, so that $A_1 = 4$ and $B_1 = 2$ are both divisible by $2 = 2^1$, so the result is true for $n = 1$. Now suppose the result is true for some $n \geq 1$, then

$$\begin{aligned} A_{n+1} + \sqrt{3}B_{n+1} &= \left(1+\sqrt{3}\right)^{2(n+1)} \\ &= \left(1+\sqrt{3}\right)^2 \left(1+\sqrt{3}\right)^{2n} \\ &= \left(4+2\sqrt{3}\right)\left(A_n + \sqrt{3}B_n\right) \end{aligned}$$

$$= 4A_n + 6B_n + \sqrt{3}(2A_n + 4B_n),$$

so that $A_{n+1} = 2(2A_n + 3B_n)$ and $B_{n+1} = 2(A_n + 2B_n)$. Therefore, from the induction hypothesis, A_{n+1} and B_{n+1} are divisible by 2^{n+1}, and the result is true for $n+1$. By the principle of mathematical induction, the result is true for all $n \geq 1$.

Therefore,

$$2A_n = \left\lceil \left(\sqrt{3}+1\right)^{2n} \right\rceil$$

is divisible by 2^{n+1} for all $n \geq 1$.

$\square$

Example 2. Let a_n, $n \geq 1$, be the solution to the discrete initial value problem

$$\begin{aligned} a_{n+1} &= 1 + \frac{n}{a_n}, \quad n \geq 1 \\ a_1 &= 1. \end{aligned}$$

(a) Show that

$$\sqrt{n} \leq a_n \leq \sqrt{n} + 1$$

for $n \geq 1$, and the inequalities are strict for all integers $n > 1$.

(b) Show that

$$\lim_{n\to\infty} \left(a_n - \sqrt{n}\right) = \frac{1}{2}.$$

Solution.

(a) We will prove that $\sqrt{n} \leq a_n \leq \sqrt{n} + 1$ for $n \geq 1$ by induction on n. If $n = 1$, then

$$\sqrt{1} = 1 = a_1 \leq 2 = \sqrt{1} + 1$$

and the result is true for $n = 1$. Now suppose the result is true for some $n \geq 1$, then

$$\begin{aligned} a_{n+1} - \sqrt{n+1} &= 1 + \frac{n}{a_n} - \sqrt{n+1} \\ &= \frac{n}{a_n} - \left(\sqrt{n+1} - 1\right) \\ &= \frac{n}{a_n} - \frac{n}{\sqrt{n+1}+1} \\ &= \frac{n\left(\sqrt{n+1} + 1 - a_n\right)}{a_n\left(\sqrt{n+1}+1\right)} \\ &\geq \frac{n\left(\sqrt{n+1} - \sqrt{n}\right)}{a_n\left(\sqrt{n+1}+1\right)} \\ &> 0 \end{aligned}$$

from the induction hypothesis, that is, $a_{n+1} > \sqrt{n+1}$. Also,

$$\begin{aligned} \sqrt{n+1} + 1 - a_{n+1} &= \sqrt{n+1} - \frac{n}{a_n} \\ &\geq \sqrt{n+1} - \frac{n}{\sqrt{n}} \\ &= \sqrt{n+1} - \sqrt{n} \\ &> 0 \end{aligned}$$

from the induction hypothesis, that is, $a_{n+1} < \sqrt{n+1} + 1$. Thus,

$$\sqrt{n+1} < a_{n+1} < \sqrt{n+1} + 1,$$

so we have shown that if the result is true for some $n \geq 1$, this implies that it is true for $n + 1$ also. Therefore, by the principle of mathematical induction, the result is true for all $n \geq 1$.

(b) In order to show that $\lim_{n\to\infty}(a_n - \sqrt{n})$ exists, we need to find a better estimate than that given in part (a). First note that since a_n satisifies

$$a_{n+1} = 1 + \frac{n}{a_n},$$

then a better estimate to a_n may be γ_n, the positive solution to the equation

$$\gamma_n = 1 + \frac{n}{\gamma_n},$$

that is, the positive solution to the quadratic equation

$$\gamma_n^2 = \gamma_n + n,$$

which is given by

$$\gamma_n = \tfrac{1}{2} + \sqrt{n + \tfrac{1}{4}}$$

for $n \geq 1$.

We will show, in fact, that

$$\gamma_{n-1} \leq a_n \leq \gamma_n \tag{$*$}$$

for all $n \geq 1$, where we define $\gamma_0 = \frac{1}{2}$; the proof is by induction on n.

For $n = 1$, we have

$$\gamma_0 = \tfrac{1}{2} < 1 = a_1 \leq \tfrac{1}{2} + \sqrt{1 + \tfrac{1}{4}} = \gamma_1,$$

and $(*)$ is true for $n = 1$. Now assume that $(*)$ is true for some $n \geq 1$, then

$$a_{n+1} = 1 + \frac{n}{a_n} \geq 1 + \frac{n}{\gamma_n} = \gamma_n,$$

and

$$a_{n+1} = 1 + \frac{n}{a_n} \leq 1 + \frac{n}{\gamma_{n-1}}.$$

Now, since $1 + \dfrac{n}{\gamma_{n-1}} = \dfrac{1}{\gamma_{n-1}} + \gamma_{n-1}$, then all we need to show is that

$$\frac{1}{\gamma_{n-1}} + \gamma_{n-1} \leq \gamma_{n+1}.$$

For $n \geq 1$, we have

$$\sqrt{n - 1 + \tfrac{1}{4}} \geq \tfrac{1}{2},$$

so that

$$2\sqrt{n - 1 + \tfrac{1}{4}} \geq 1,$$

and

$$\left(1 + \sqrt{n - 1 + \tfrac{1}{4}}\right)^2 = 1 + 2\sqrt{n - 1 + \tfrac{1}{4}} + n - 1 + \tfrac{1}{4} \geq n + 1 + \tfrac{1}{4}.$$

Taking the nonnegative square root of both sides of this inequality, we have

$$1 + \sqrt{n - 1 + \tfrac{1}{4}} \geq \sqrt{n + 1 + \tfrac{1}{4}},$$

so that

$$2\left(\tfrac{1}{2} + \sqrt{n - 1 + \tfrac{1}{4}}\right) \geq \sqrt{n + 1 + \tfrac{1}{4}} + \sqrt{n - 1 + \tfrac{1}{4}}.$$

Thus,

$$\frac{2}{\sqrt{n+1+\frac{1}{4}}+\sqrt{n-1+\frac{1}{4}}} \geq \frac{1}{\frac{1}{2}+\sqrt{n-1+\frac{1}{4}}},$$

that is,

$$\sqrt{n+1+\frac{1}{4}}-\sqrt{n-1+\frac{1}{4}} \geq \frac{1}{\frac{1}{2}+\sqrt{n-1+\frac{1}{4}}},$$

and thus

$$\gamma_{n+1}-\gamma_{n-1} \geq \frac{1}{\gamma_{n-1}},$$

for all $n \geq 1$, which is what we wanted to show. Therefore, $a_{n+1} \leq \gamma_{n+1}$, and $(*)$ is also true for $n+1$. Therefore, by the principle of mathematical induction $(*)$ is true for all $n \geq 1$.

Finally, using $(*)$ we have

$$-\frac{\sqrt{4n}-\sqrt{4(n-1)+1}}{2} \leq a_n - \sqrt{n} - \tfrac{1}{2} \leq \frac{\sqrt{4n+1}-\sqrt{4n}}{2},$$

but

$$\lim_{n\to\infty} \frac{\sqrt{4n}-\sqrt{4(n-1)+1}}{2} = \lim_{n\to\infty} \frac{3}{2\left(\sqrt{4n}+\sqrt{4(n-1)+1}\right)} = 0$$

and

$$\lim_{n\to\infty} \frac{\sqrt{4n+1}-\sqrt{4n}}{2} = \lim_{n\to\infty} \frac{1}{2\left(\sqrt{4n+1}+\sqrt{4n}\right)} = 0,$$

so that $\lim\limits_{n\to\infty} (a_n - \sqrt{n}) = \frac{1}{2}$.

$\square$