Computing Science 204 (B1)

Generating Function for Fibonacci Numbers with Even Index
Thursday, July 13

Let $b_n = F_{2n}$ for $n \geq 0$, be the Fibonacci numbers with even index.

(a) Find a homogeneous, linear, second order difference equation with constant coefficients satisfied by the sequence $\{b_n\}_{n\geq 0}$, together with two initial conditions for b_0 and b_1.

(b) Find the generating function

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

for the sequence $\{b_n\}_{n\geq 0}$.

Solution:

(a) From the definition we have

$$F_{2n+2} = F_{2n+1} + F_{2n} \qquad \text{and} \qquad F_{2n+1} = F_{2n} + F_{2n-1},$$

and therefore,

$$F_{2n+2} = 2F_{2n} + F_{2n-1}.$$

Similarly,

$$F_{2n-1} = F_{2n} - F_{2n-2}.$$

Therefore,

$$F_{2n+2} = 3F_{2n} - F_{2n-2}.$$

Now let $b_n = F_{2n}$, then

$$b_{n+1} = 3b_n - b_{n-1}$$

for all $n \geq 1$. Also, $b_0 = F_0 = 0$ and $b_1 = F_2 = 1$, so that b_n satisfies the discrete initial value problem

$$\begin{aligned} b_{n+2} &= 3b_{n+1} - b_n, \quad n \geq 0 \\ b_0 &= 0 \\ b_1 &= 1. \end{aligned}$$

(b) If we multiply both sides of the recurrence relation for b_n by x^{n+2} and sum over n, then we get

$$\sum_{n=0}^{\infty} b_{n+2}x^{n+2} = x \cdot \sum_{n=0}^{\infty} 3b_{n+1}x^{n+1} - x^2 \cdot \sum_{n=0}^{\infty} b_n x^n,$$

so that

$$B(x) - b_0 - b_1 x = 3x(B(x) - b_0) - x^2 B(x),$$

that is,

$$(1 - 3x + x^2)B(x) = b_0 + b_1 x - 3xb_0 = x,$$

so that

$$B(x) = \frac{x}{1 - 3x + x^2}.$$