Computing Science 204 (B1)
Systems of Linear Difference Equations
Monday, July 24

Example. *Systems of Linear Difference Equations*
Define

$$a_n = \sum_{k=0}^{n} \binom{n+k}{2k} \qquad \text{for} \quad n \geq 1$$
$$b_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \qquad \text{for} \quad n \geq 1$$

with $a_0 = 1$ and $b_0 = 0$.

(a) Show that a_n and b_n satisfy the recurrence relations

$$a_{n+1} = a_n + b_{n+1} \qquad \text{for} \quad n \geq 0$$
$$b_{n+1} = a_n + b_n \qquad \text{for} \quad n \geq 0$$

(b) Express a_n and b_n in terms of the Fibonacci numbers.

proof. We will need the following addition formula for the binomial coefficients

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

for all $n \geq 1$ and $1 \leq r \leq n$.

(a) If $n \geq 1$, then

$$\begin{aligned}
a_n + b_n &= \sum_{k=0}^{n} \binom{n+k}{2k} + \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \\
&= \binom{n}{0} + \sum_{k=1}^{n-1} \binom{n+k}{2k} + \binom{2n}{2n} + \binom{n}{1} + \sum_{k=1}^{n-1} \binom{n+k}{2k+1} \\
&= \binom{n}{0} + \binom{n}{1} + \sum_{k=1}^{n-1} \left\{ \binom{n+k}{2k} + \binom{n+k}{2k+1} \right\} + \binom{2n}{2n} \\
&= \binom{n}{0} + \binom{n}{1} + \sum_{k=1}^{n-1} \binom{n+1+k}{2k+1} + \binom{2n+1}{2n+1} \\
&= \binom{n+1}{1} + \sum_{k=1}^{n} \binom{n+1+k}{2k+1} \\
&= \sum_{k=0}^{n} \binom{n+1+k}{2k+1} \\
&= b_{n+1}.
\end{aligned}$$

Therefore, $b_{n+1} = a_n + b_n$ for all $n \geq 1$, and since $a_0 = 1$ and $b_0 = 0$, then $b_1 = 1 = a_0 + b_0$, so that $b_{n+1} = a_n + b_n$ for all $n \geq 0$.

If $n \geq 1$, then

$$\begin{aligned}
a_n + b_{n+1} &= \sum_{k=0}^{n} \binom{n+k}{2k} + \sum_{k=0}^{n} \binom{n+1+k}{2k+1} \\
&= \binom{n}{0} + \sum_{k=1}^{n} \binom{n+1+k-1}{2(k-1)+2} + \sum_{k=0}^{n} \binom{n+1+k}{2k+1} \\
&= \binom{n}{0} + \sum_{\ell=0}^{n-1} \binom{n+1+\ell}{2\ell+2} + \sum_{k=0}^{n} \binom{n+1+k}{2k+1} \\
&= \binom{n}{0} + \binom{2n+1}{2n+1} + \sum_{k=0}^{n-1} \left\{ \binom{n+1+k}{2k+2} + \binom{n+1+k}{2k+1} \right\} \\
&= \binom{n}{0} + \binom{2n+1}{2n+1} + \sum_{k=0}^{n-1} \binom{n+2+k}{2k+2} \\
&= \binom{n}{0} + \binom{2n+1}{2n+1} + \sum_{k=1}^{n} \binom{n+1+k}{2k} \\
&= \sum_{k=0}^{n} \binom{n+1+k}{2k} + \binom{2n+2}{2n+2} \\
&= \sum_{k=0}^{n+1} \binom{n+1+k}{2k} \\
&= a_{n+1}.
\end{aligned}$$

Therefore, $a_{n+1} = a_n + b_{n+1}$ for all $n \geq 1$, and since $a_0 = 1$ and $b_1 = 1$, then $a_1 = 2 = a_0 + b_1$, and $a_{n+1} = a_n + b_{n+1}$ for all $n \geq 0$.

(b) Replacing n by $n+1$ in the first recurrence relation, we have

$$a_{n+2} = a_{n+1} + b_{n+2} = a_{n+1} + a_{n+1} + b_{n+1} = 2a_{n+1} + (a_{n+1} - a_n) = 3a_{n+1} - a_n.$$

Therefore, the sequence $\{a_n\}_{n \geq 0}$ satisfies the initial value problem

$$\begin{aligned}
a_{n+2} &= 3a_{n+1} - a_n \qquad n \geq 0 \\
a_0 &= 1 \\
a_1 &= 2.
\end{aligned}$$

We can use this recurrence to generate the rest of the sequence:

$$a_2 = 5,\ a_3 = 13,\ a_4 = 34,\ a_5 = 89,\ a_6 = 233,\ \cdots$$

and an easy induction argument shows that $a_n = \sum\limits_{k=0}^{n} \binom{n+k}{2k} = F_{2n+1}$ for all $n \geq 0$.

Similarly, the sequence $\{b_n\}_{n \geq 0}$ satisfies the intial value problem

$$\begin{aligned}
b_{n+2} &= 3b_{n+1} - b_n \qquad n \geq 0 \\
b_0 &= 0 \\
b_1 &= 1.
\end{aligned}$$

We can use this recurrence to generate the rest of the sequence:

$$b_2 = 3,\ b_3 = 8,\ b_4 = 21,\ b_5 = 55,\ b_6 = 144,\ \cdots$$

and an easy induction argument shows that $b_n = \sum\limits_{k=0}^{n-1} \binom{n+k}{2k+1} = F_{2n}$ for all $n \geq 0$.