Computing Science 204 (B1)

Some Examples of DFS and BFS Trees

Friday, August 11

Example 1. Let v be a vertex in a graph G. Calculate the height of the DFS and BFS trees in G rooted at v for the following graphs:

(a) K_n, the complete graph with n vertices.

(b) C_n, the cycle with n vertices.

Solution.

(a) Let $K_n = (V, E)$ be the complete graph on n vertices. We may assume that

$$V = \{1, 2, 3, \ldots, n\} \qquad \text{and} \qquad E = \big\{\{i, j\} \mid i \neq j, 1 \leq i, j \leq n\big\}.$$

Given any DFS spanning tree for K_n, without loss of generality we may assume that 1 is the root (otherwise just renumber the vertices). Since there is an edge between any two distinct vertices, we may assume that the DFS algorithm visits the nodes in the order, $1 \to 2 \to 3 \to \cdots \to n$ and does not have to back up from a vertex until it visits n. At that point all the vertices in K_n will have been marked as visited, and the algorithm returns. The height of any DFS spanning tree for K_n is therefore $n - 1$.

Given any BFS spanning tree for K_n, assuming as above that 1 is the root, in this case, since 1 is adjacent to all of the other $n-1$ vertices in K_n, the BFS algorithm will visit all $n-1$ vertices adjacent to 1 before it moves on to the next level. Here though, there are no vertices a distance 2 from the root, and therefore the height of any BFS spanning tree for K_n is 1.

(b) Let $C_n = (V, E)$ be the cycle graph with n vertices. Again, we may assume without loss of generality that

$$V = \{1, 2, 3, \ldots, n\},$$

and, in this case, that

$$E = \big\{\{i, i+1\} \mid 1 \leq i \leq n-1\big\} \cup \big\{\{1, n\}\big\}.$$

Given any DFS spanning tree for C_n, we may assume as above that 1 is the root and that the DFS algorithm visits the vertices in the order $1 \to 2 \to 3 \to \cdots \to n$ and does not have to back up from a vertex until it visits n, as before, at which point all the vertices have been marked as visited, and the algorithm returns. Therefore, the height of any DFS spanning tree for C_n is also $n - 1$.

Given any BFS spanning tree for C_n, assuming again that 1 is the root, we consider separately the cases when n is even or odd.

(i) If $n = 2k + 1$, for some positive integer k, then in the BFS tree, the root 1 is at level 0, and for each i, with $2 \leq i \leq k+1$, vertices i and $2k - i + 3$ are at level $i - 1$. Therefore, the height of the BFS tree is $k = \lfloor \frac{n}{2} \rfloor$.

(ii) If $n = 2k$, for some positive integer k, then in the BFS tree, the root 1 is at level 0 as before, and for each i, with $2 \leq i \leq k$, vertices i and $2k - i + 2$ are at level $i - 1$, but now however, vertex $k + 1$ is at level k. Therefore, the height of the BFS tree is $k = \lfloor \frac{n}{2} \rfloor$.

Thus, in both cases, the height of any BFS spanning tree for the cycle graph C_n is $\lfloor \frac{n}{2} \rfloor$.

$\square$

Example 2. Let G be a connected graph which is regular, that is, each vertex has the same degree, k, and let d be the height of the BFS tree rooted at a vertex of G. Show that the number of vertices of G is at most

$$1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{d-1}.$$

Solution. Here $G = (V, E)$ is a connected graph which is regular with $\deg(v) = k$ for each $v \in V$. In any BFS spanning tree for G, the root, which is at level 0, will have exactly k children. If v is a vertex at level $i \geq 1$, then v can have at most $k - 1$ children (it has one edge connecting it to its parent). We have the following estimates

$$\begin{array}{lcl}
\#\text{ vertices at level } 0 & = & 1 \\
\#\text{ vertices at level } 1 & = & k \\
\#\text{ vertices at level } 2 & \leq & k(k-1) \\
\#\text{ vertices at level } 3 & \leq & k(k-1)^2 \\
 & \vdots & \\
\#\text{ vertices at level } i & \leq & k(k-1)^{i-1} \\
 & \vdots & \\
\#\text{ vertices at level } d & \leq & k(k-1)^{d-1}.
\end{array}$$

The number of vertices of G is therefore at most

$$1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{d-1},$$

as required.

$\square$