Computing Science 204 (B1)
Some Examples of Difference Equations
Friday, July 21

Example 1. Consider the initial value problem

$$\begin{aligned} t_n - t_{n-1} &= n \cdot n! \qquad n \geq 1 \\ t_0 &= 0. \end{aligned}$$

(a) Find an exact solution to this problem.
(b) Show that the solution satisfies $t_{n-1} \in O(n^n)$ and $t_{n-1} \in \Omega(n^{\frac{n}{2}})$.

proof.

(a) If we sum this relation for values of $k = 1, 2, \ldots, n$, then we get

$$\begin{aligned} t_n - t_0 &= \sum_{k=1}^{n} (t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} k \cdot k! \end{aligned}$$

and since $t_0 = 0$, then

$$t_n = \sum_{k=1}^{n} k \cdot k! = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!.$$

If we examine the first few terms, a pattern emerges

$$\begin{aligned} &1 \cdot 1! = 1 = 2 - 1 = 2! - 1 \\ &1 \cdot 1! + 2 \cdot 2! = 5 = 6 - 1 = 3! - 1 \\ &1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! = 5 + 18 = 23 = 24 - 1 = 4! - 1 \\ &1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! = 23 + 96 = 119 = 120 - 1 = 5! - 1 \\ &1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! = 119 + 600 = 719 = 720 - 1 = 6! - 1 \\ &\qquad\vdots \end{aligned}$$

and an easy induction argument establishes that

$$t_n = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

for all $n \geq 1$.

(b) For $n \geq 1$, we have $t_{n-1} = n! - 1$, and clearly

$$n! - 1 < n! < n^n$$

for all $n \geq 1$, so that $t_{n-1} \in O(n^n)$. On the other hand,

$$\begin{aligned} (n!)^2 &= n \cdot 1 \cdot (n-1) \cdot 2 \cdot (n-2) \cdot 3 \cdots 2 \cdot (n-1) \cdot n \cdot 1 \\ &= \prod_{k=1}^{n} k \cdot (n - k + 1). \end{aligned}$$

On the interval $1 \leq k \leq n$, the quadratic function $f(k) = k \cdot (n - k + 1)$ attains its maximum value of $f_{\max} = n$ at $k = 1$ and $k = n$, so that

$$(n!)^2 \geq n \cdot n \cdots n = n^n$$

and therefore $n! > n^{\frac{n}{2}}$ for all $n \geq 1$, so that $t_{n-1} \in \Omega(n^{\frac{n}{2}})$.

Example 2. Let a and b be fixed positive numbers, and consider the initial value problem

$$\begin{aligned} a_{n+2} &= (a_n\, a_{n+1})^{\frac{1}{2}}, \qquad n \geq 0 \\ a_0 &= a \\ a_1 &= b. \end{aligned}$$

(a) Find the exact solution to this initial value problem.

(b) Does $\gamma = \lim_{n\to\infty} a_n$ exist? If so, find it.

proof.

(a) Since $a_0 = a > 0$ and $a_1 = b > 0$, then an easy induction argument shows that $a_n > 0$ for all $n \geq 0$. Thus, we can define $b_n = \log a_n$, and taking the logarithm of both sides of the recurrence relation, we see that b_n satisfies

$$b_{n+1} = \frac{1}{2} b_n + \frac{1}{2} b_{n-1},$$

which is a linear, second order, homogeneous, difference equation with constant coefficients. Assuming a solution of the form $b_n = \lambda^n$, the characteristic equation is $\lambda^2 = \frac{1}{2}\lambda + \frac{1}{2}$, with distinct real solutions $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$. The general solution to this equation is

$$b_n = A \cdot 1^n + B \cdot \frac{(-1)^n}{2^n} = A + B \cdot \frac{(-1)^n}{2^n},$$

and we can determine the initial conditions from $b_0 = \log a$ and $b_1 = \log b$. We get the system of linear equations

$$\begin{aligned} A + B &= \log a \\ A - \frac{1}{2} \cdot B &= \log b, \end{aligned}$$

with unique solution $A = \log \left(a\, b^2\right)^{\frac{1}{3}}$ and $B = \log \left(\frac{a}{b}\right)^{\frac{2}{3}}$. Therefore,

$$b_n = \log \left(a\, b^2\right)^{\frac{1}{3}} + \frac{(-1)^n}{2^n} \cdot \log \left(\frac{a}{b}\right)^{\frac{2}{3}} = \log \left[\left(a\, b^2\right)^{\frac{1}{3}} \left(\frac{a}{b}\right)^{\frac{2}{3} \cdot \frac{(-1)^n}{2^n}} \right],$$

and

$$a_n = \left(a\, b^2\right)^{\frac{1}{3}} \left(\frac{a}{b}\right)^{\frac{2}{3} \cdot \frac{(-1)^n}{2^n}}$$

for $n \geq 0$.

(b) Since

$$\lim_{n\to\infty} \frac{(-1)^n}{2^n} = 0$$

and the exponential function is continuous, then the limit $\gamma = \lim_{n\to\infty} a_n$ exists and $\gamma = \left(a\, b^2\right)^{\frac{1}{3}}$.

Example 3. For an integer $n \geq 1$, let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be the n^{th} Harmonic number, and let p and q be fixed integers with $p \geq q \geq 1$, and let

$$a_n = \sum_{k=q\cdot n+1}^{p\cdot n} \frac{1}{k} \qquad \text{and} \qquad b_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}.$$

(a) Find $\lim\limits_{n\to\infty} a_n$.

(b) Show that if $q = 1$ and $p = 2$, then $b_{2n} = a_n$.

(c) Use part (b) to find $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n}$.

proof.

(a) Note that

$$\begin{aligned}
a_n &= \sum_{k=q\cdot n+1}^{p\cdot n} \frac{1}{k} \\
&= H_{p\cdot n} - H_{q\cdot n} \\
&= H_{p\cdot n} - \log(p \cdot n) - \big(H_{q\cdot n} - \log(q \cdot n)\big) + \log(p \cdot n) - \log(q \cdot n) \\
&= \gamma_{p\cdot n} - \gamma_{q\cdot n} + \log\left(\frac{p}{q}\right)
\end{aligned}$$

and since $\lim\limits_{n\to\infty} \gamma_n = \lim\limits_{n\to\infty} (H_n - \log n) = \gamma$, the Euler–Mascheroni constant, then $\lim\limits_{n\to\infty} a_n$ exists and $\lim\limits_{n\to\infty} a_n = \log\left(\frac{p}{q}\right)$.

(b) Now, taking $p = 2$ and $q = 1$, we have

$$\begin{aligned}
H_{2n} &= \sum_{\substack{k=1 \\ k \text{ even}}}^{2n} \frac{1}{k} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{2n} \frac{1}{k} \\
&= \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{2n} \frac{1}{k} \\
&= \frac{1}{2} H_n + \sum_{\substack{k=1 \\ k \text{ odd}}}^{2n} \frac{1}{k}
\end{aligned}$$

so that

$$a_n = H_{2n} - H_n = -\frac{1}{2} H_n + \sum_{\substack{k=1 \\ k \text{ odd}}}^{2n} \frac{1}{k} = -\sum_{\substack{k=1 \\ k \text{ even}}}^{2n} \frac{1}{k} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{2n} \frac{1}{k} = \sum_{k=1}^{2n} \frac{(-1)^{n-1}}{k},$$

so that $a_n = b_{2n}$. From the previous result, that is, part (a), we have $\lim\limits_{n\to\infty} a_n = \sum\limits_{k=1}^{\infty} \dfrac{(-1)^n}{k} = \log 2$.

Example 4. Code words from the alphabet $\{0, 1, 2, 3\}$ are to be recognized as *legitimate* if and only if they have an even number of 0's. How many legitimate code words of length n are there?

proof. For $n \geq 1$, let a_n be the number of legitimate code words of length n, and note that the number of illegitimate code words of length n is $4^n - a_n$.

There are $3 \cdot a_{n-1}$ legitimate code words of length n that start with 1, 2, or 3, and there are $4^{n-1} - a_{n-1}$ legitimate code words that start with a 0. Therefore, $a_n = 3\,a_{n-1} + 4^{n-1} - a_{n-1}$ for $n \geq 2$, while $a_1 = 3$. Thus, a_n satisfies the initial value problem

$$\begin{aligned} a_n &= 2\,a_{n-1} + 4^{n-1} \quad n \geq 2 \\ a_1 &= 3 \end{aligned}$$

which is a first order linear nonhomogeneous difference equation with constant coefficients.

Using the method of characteristic equations, in order to find the general solution to the homogeneous equation

$$t_n - 2t_{n-1} = 0 \tag{$*$}$$

we assume a solution of the form $a_n = \lambda^n$, and substitute this into the homogeneos equation $(*)$, to get

$$\lambda^n - 2\lambda^{n-1} = 0,$$

and dividing by λ^{n-1}, we get the characteristic equation

$$\lambda - 2 = 0,$$

with solution $\lambda = 2$. Therefore, the general solution to $(*)$ is

$$h_n = A \cdot 2^n$$

where A is a constant.

Using the annihilator method to find a particular solution to the nonhomogeneous equation, since the right hand side is 4^{n-1}, our solution is $s_n = B \cdot 4^{n-1}$, and substituting this into the original equation, we get $B = 2$, so that $s_n = 2 \cdot 4^{n-1}$, and the general solution to the nonhomogeneous equation is

$$a_n = A \cdot 2^n + \frac{1}{2} \cdot 4^n$$

for $n \geq 1$. The constant A is determined from the initial condition $a_1 = 3 = 2\,A + 2$ so that $A = \dfrac{1}{2}$.

Therefore, the number of legitimate code words of length n is

$$a_n = \frac{1}{2} \cdot \left(2^n + 4^n\right)$$

for $n \geq 1$.