Computing Science 204 (B1)
Characteristic Equation with Complex Roots
Tuesday, July 25

Example. Consider the discrete initial value problem

$$\begin{aligned} a_{n+2} - 2a_{n+1} + 2a_n = 0 \qquad & n \geq 0 \\ a_0 = 0 & \\ a_1 = 1. & \end{aligned}$$

Find the unique solution to this problem and use it to prove the identity

$$2^{\frac{n}{2}} \sin \tfrac{n\pi}{4} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{2k+1}.$$

proof. The characteristic equation of this homogeneous, linear, constant coefficient, second-order difference equation is

$$\lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0,$$

with the complex roots

$$\lambda_1 = 1 + i \qquad \text{and} \qquad \lambda_2 = 1 - i,$$

where $i^2 = -1$.

The general solution to the difference equation is

$$a_n = A \cdot (1+i)^n + B \cdot (1-i)^n$$

where the constants A and B are obtained from the initial conditions as follows:

$$\begin{aligned} a_0 &= 0 = A + B \\ a_1 &= 1 = 2iA \end{aligned}$$

so that $A = -B = \dfrac{1}{2i}$. Therefore, the unique solution to the initial value problem is

$$a_n = \frac{1}{2i}\left[(1+i)^n - (1-i)^n\right]$$

for $n \geq 0$.

From the binomial theorem we have

$$(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k$$

and

$$(1-i)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k i^k,$$

so that

$$(1+i)^n - (1-i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k \left[1 - (-1)^k\right],$$

and the only terms that survive are the terms when k is odd.

Now if $k = 2m+1$, then

$$i^{2m+1} = i \cdot i^{2m} = i(-1)^m,$$

and since we want $2m+1 \leq n$, then we must have $m \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, and

$$(1+i)^n - (1-i)^n = 2i \sum_{m=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2m+1} (-1)^m,$$

and therefore,

$$a_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k+1} (-1)^k$$

for $n \geq 0$.

Also,

$$\begin{aligned} a_n &= \frac{1}{2i}\left[(1+i)^n - (1-i)^n\right] \\ &= \frac{1}{2i}\left[\left(\tfrac{1+i}{\sqrt{2}}\right)^n - \left(\tfrac{1-i}{\sqrt{2}}\right)^n\right] \cdot 2^{\frac{n}{2}} \\ &= \frac{1}{2i}\left[e^{\frac{in\pi}{4}} - e^{-\frac{in\pi}{4}}\right] \cdot 2^{\frac{n}{2}} \\ &= 2^{\frac{n}{2}} \sin \tfrac{n\pi}{4}, \end{aligned}$$

and therefore,

$$2^{\frac{n}{2}} \sin \tfrac{n\pi}{4} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n}{2k+1}$$

for $n \geq 0$.

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Note: We have used Euler's formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{and} \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

to write

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

in the above.