Computing Science 204 (B1)
The Binomial Series
Thursday, July 20

The following expansion is known as the *binomial series*:

Theorem. For $-1 < x < 1$,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

where α is an arbitrary real number, and

$$\binom{\alpha}{n} = \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, & \text{for} \quad n \geq 1, \\ 1, & \text{for} \quad n = 0. \end{cases}$$

proof. First we show that the series converges for all x in the open interval $-1 < x < 1$. For $n \geq 0$, let

$$a_n = \left| \binom{\alpha}{n} x^n \right|,$$

then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} x \right| = \lim_{n\to\infty} \frac{n-\alpha}{n+1} |x| = |x|,$$

and by the ratio test, the series converges absolutely for $|x| < 1$.

Now define the function g on the interval $-1 < x < 1$ as follows,

$$g(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad -1 < x < 1,$$

differentiating, we have

$$g'(x) = \sum_{n=1}^{\infty} n \binom{\alpha}{n} x^{n-1} = \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} x^{n-1},$$

so that

$$\begin{aligned} (1+x)g'(x) &= \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} x^{n-1} + \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} x^n \\ &= \alpha \sum_{n=0}^{\infty} \binom{\alpha-1}{n} x^n + \alpha \sum_{n=1}^{\infty} \binom{\alpha-1}{n-1} x^n \\ &= \alpha \left[1 + \sum_{n=1}^{\infty} \left\{ \binom{\alpha-1}{n} + \binom{\alpha-1}{n-1} \right\} x^n \right] \\ &= \alpha \left[1 + \sum_{n=1}^{\infty} \binom{\alpha}{n} x^n \right] \\ &= \alpha g(x), \end{aligned}$$

that is, $g'(x) = \dfrac{\alpha g(x)}{1+x}$, for $-1 < x < 1$.

Now define $h(x) = \dfrac{g(x)}{(1+x)^\alpha}$ for $-1 < x < 1$, then

$$h'(x) = \frac{g'(x)}{(1+x)^\alpha} - \frac{\alpha g(x)}{(1+x)^{\alpha+1}} = \frac{\alpha g(x)}{(1+x)^{\alpha+1}} - \frac{\alpha g(x)}{(1+x)^{\alpha+1}} = 0$$

for all $-1 < x < 1$, so that $h(x)$ is constant on the interval $-1 < x < 1$. Since $h(0) = 1$, then $h(x) = 1$ for all $x \in (-1, 1)$, that is,

$$(1+x)^\alpha = g(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

for $-1 < x < 1$.

$\square$