Computing Science 204 (B1)

Binet's Formula for the Fibonacci Numbers

(a) Let $\{a_n\}_{n\geq 0}$, be the unique solution to the discrete initial value problem

$$\begin{aligned} a_{n+2} &= a_{n+1} + a_n \qquad n \geq 0 \\ a_0 &= 0 \\ a_1 &= 1, \end{aligned}$$

that is, a_n is the n^{th} term in the Fibonacci sequence, then

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all $n \geq 0$. This is called *Binet's formula*, is was first discovered by DeMoivre, and later independently by Binet.

(b) Let $\alpha = \dfrac{1+\sqrt{5}}{2}$, then a_n is the nearest integer to $\dfrac{\alpha^n}{\sqrt{5}}$ for all $n \geq 0$.

SOLUTION:

(a) Assuming a solution of the form $a_n = \lambda^n$, the characteristic equation becomes $\lambda^2 = \lambda + 1$, with two distinct real roots,

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2},$$

so the general solution is

$$a_n = A\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n + B\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where the constants A and B are determined from the initial conditions

$$a_0 = A + B = 0 \qquad \text{and} \qquad a_1 = A\cdot\left(\frac{1+\sqrt{5}}{2}\right) + B\cdot\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

to be

$$A = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad B = -\frac{1}{\sqrt{5}},$$

so that

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all $n \geq 0$.

(b) Since $\dfrac{1}{\sqrt{5}} < \dfrac{1}{2}$, then

$$\left|\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right| < \frac{1}{2},$$

so that

$$-\frac{1}{2} < -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n < \frac{1}{2}$$

and therefore, from Binet's formula, $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \dfrac{1}{2} < a_n < \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n + \dfrac{1}{2}$. Equivalently, $\left|a_n - \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n\right| < \dfrac{1}{2}$ so that a_n is the nearest integer to $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$.