CMPUT 204 (B1) – Assignment #1

DUE DATE: July 19, 2000

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1. *Insertion Sort*

   A recursive version of the sorting algorithm *InsertionSort* is given below.

```
void InsertionSort(int *L, int m, int n)
{
    if (m > 0)
    {
        int j, k = m, temp = L[m-1];
        while ((k < n) && (L[k] < temp))
            k++;
        j = k-1;
        for (k = m; k <= j; k++)
            L[k-1] = L[k];
        L[j] = temp;
        InsertionSort(L,m-1,n);
    }
}
```

   In order to sort a list `L` with `n` entries, the procedure is called as `InsertionSort(L,n,n)`.

   (a) Explain briefly how the algorithm works.

   (b) What is a worst–case input for the algorithm? That is, for what input would the algorithm make the most number of comparisons of list entries?

   (c) Let $T(n)$ be the number of comparisons of list entries made in the worst case when *InsertionSort* is performed on a list L containing n entries. Find a recurrence relation together with an initial condition satisfied by $T(n)$ for $n \geq 1$.

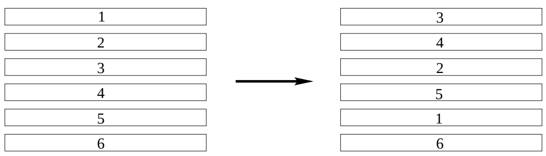
   (d) Solve this recurrence relation to get $T(n)$ for $n \geq 1$.

2. *Card Shuffling*

   (a) Given an ordinary deck of playing cards, arranged in any order, how many shuffles are required before the deck returns to its original order? To idealize the real life situation, we stipulate that only the *modified perfect faro shuffle* is permitted, that is, cut the deck into two equal 26-card packs; then proceed by alternating cards from each pack. The cards formerly in positions 1, 2, ..., 26 are moved to positions 2, 4, ..., 52; and the cards formerly in positions 27, 28, ..., 52 are moved to positions 1, 3, ..., 51. Thus, if a particular card starts in position x, it will end in position y, where $1 \leq y \leq 52$ and $2x \equiv y \mod 53$. After n shuffles, this particular card will be in position w, where $1 \leq w \leq 52$ and $2^n x \equiv w \mod 53$.

   More generally, if we have a deck of m cards, where m is an even integer, find the smallest integer n such that n shuffles will return the cards to the original order. What is n for $m = 6$? $m = 8$? $m = 12$? $m = 62$?

   (b) Suppose that instead of performing a modified perfect faro shuffle as described above, we shuffle as follows: Take the bottom and top cards of the deck and place them on the table to start a new deck. Then take the remaining bottom and top cards and place them on the newly started pile. Continue this process until all cards are gone from the original deck. For example, if the original deck has six cards, then we shuffle them as shown in the figure.



   Show that the cards in a deck of 2^n cards will return to their original positions after $n+1$ such shuffles.

3. *Monotone Subsequences.*

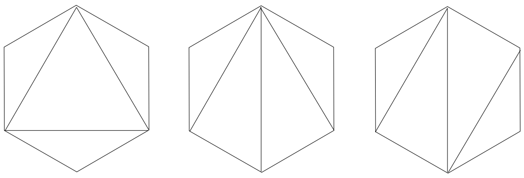
(a) Given a sequence $a_1,\ a_2,\ a_3,\ \ldots\ a_{n^2+1}$ of any n^2+1 different positive integers, show that there is either an increasing subsequence of $n+1$ terms, or else there is a decreasing sequence of $n+1$ terms.

For example, if $n = 3$, then any sequence of 10 different positive integers either contains an increasing sequence of four terms or else a decreasing sequence of four terms.

(b) Show that the result in part (a) is the best possible in the sense that it is not true for any shorter original sequence, that is, if we start with only n^2 different positive integers, it is possible to have no increasing or decreasing sequence of $n+1$ terms.

4. *Number of Triangulations of a Convex n-gon.*

Let a_n be the number of ways that a convex polygon with n sides can be divided into triangles by drawing diagonals that do not intersect. (A polygon is *convex* if all its diagonals lie in the interior of the polygon.)



(a) Calculate $a_3,\ a_4,\ a_5,\ a_6$.

(b) Show that no matter how a convex polygon with n sides is triangulated, the number of diagonals is always $n-3$ and the number of triangles is always $n-2$.

(c) Define $a_2 = 1$, and show that for $n \geq 3$,

$$a_n = a_2\, a_{n-1} + a_3\, a_{n-2} + a_4\, a_{n-3} + \cdots + a_{n-1}\, a_2$$

(d) Show that the sequence $\{a_n\}_{n\geq 3}$ also satisfies the discrete initial value problem

$$\begin{aligned} (n-3)a_n &= \frac{n}{2}\left(a_3\, a_{n-1} + a_4\, a_{n-2} + \cdots + a_{n-1}\, a_3\right), \quad n \geq 4 \\ a_3 &= 1. \end{aligned}$$

(e) Solve these recurrence relations to find an explicit formula for $a_n,\ n \geq 3$.

5. *A Familiar Sequence.*

Given the discrete initial value problem

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} + 3a_{n-3} + \cdots + na_0, \quad n \geq 1 \\ a_0 &= 1, \end{aligned}$$

(a) Calculate the values of a_n for small values of n; from the observed pattern, make a conjecture as to the general term a_n.

(b) Find the generating function

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

for the sequence $\{a_n\}$, and use this generating function to prove the conjecture in part (a).