Math 334—Solutions to Assignment 4

1. Take the Laplace transform of the DE and let $Y(s) = \mathcal{L}\{y(t)\}$:

$$s^2Y - 1 + Y = \mathcal{L}\{g\}\,.$$

Then

$$\begin{aligned} Y(s) &= \frac{1}{1+s^2} + \frac{1}{s^2+1}\mathcal{L}\{g(t)\} \\ &= \frac{1}{1+s^2} + \mathcal{L}\{\sin t\}\mathcal{L}\{g(t)\}\,. \end{aligned}$$

Take the inverse transform, using the convolution theorem to handle the right-most term,

$$\begin{aligned} y(t) &= \sin t + (\sin * g)(t) \\ &= \sin t + \int_0^t \sin(t-v)g(v)\,\mathrm{d}v \\ &= \sin t + \int_0^t g(t-v)\sin v\,\mathrm{d}v\,. \end{aligned}$$

(Any of these ways of expressing the answer suffices.)

2. By the convolution theorem, note that

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^3}\cdot\frac{1}{s^2+1}\right\} &= \frac{1}{2}t^2 * \sin t \\ &= \frac{1}{2}\int_0^t v^2\sin(t-v)\,\mathrm{d}v \\ &= \frac{1}{2}\int_0^t (t-v)^2\sin v\,\mathrm{d}v \\ &= \frac{1}{2}t^2\int_0^t \sin v\,\mathrm{d}v - 2t\frac{1}{2}\int_0^t v\sin v\,\mathrm{d}v \\ &\quad +\frac{1}{2}\int_0^t v^2\sin v\,\mathrm{d}v\,. \end{aligned}$$

After computing these integrals (using formulas in the front cover of the text, or integration by parts), you should get

$$\cos t - 1 + \frac{1}{2}t^2\,.$$

3. Take the Laplace transform of the equation. Notice the integral is a convolution, so its Laplace transform is a product. You should get

$$\begin{aligned}
& Y + \mathcal{L}\{t\}Y = \mathcal{L}\{t^2\} \\
\Rightarrow\quad & Y(s) + \frac{1}{s^2}Y(s) = \frac{2}{s^3} \\
\Rightarrow\quad & Y(s) = \frac{2}{s(s^2+1)} = \frac{2}{s} - \frac{2s}{s^2+1} \\
\Rightarrow\quad & y(t) = 2 - 2\cos t\,.
\end{aligned}$$

4. (a) First, we note that

$$\mathcal{L}\{e^{-t}\delta(t-2)\} = \int_0^\infty e^{-st}e^{-t}\delta(t-2)\,\mathrm{d}t = \int_{-\infty}^\infty e^{-st}e^{-t}\delta(t-2)\,\mathrm{d}t = e^{-2(s+1)}.$$

Taking the Laplace transform of both sides of the equation, we obtain

$$s^2Y - 2s + 5 + 5sY - 10 + 6Y = e^{-2(s+1)}$$

or

$$Y = \frac{2s+5}{s^2+5s+6} + e^{-2}e^{-2s}\frac{1}{s^2+5s+6}.$$

Partial fractioning yields

$$\frac{2s+5}{s^2+5s+6} = \frac{1}{s+2} + \frac{1}{s+3}$$

and

$$\frac{1}{s^2+5s+6} = \frac{1}{s+2} - \frac{1}{s+3}.$$

Using the formula $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$ we finally obtain

$$y(t) = e^{-2t} + e^{-3t} + e^{-2}\left(e^{-2(t-2)} - e^{-3(t-2)}\right)u(t-2).$$

(b) The Laplace transforms of the two equations read

$$sX(s) - 3X(s) + 2Y(s) = \frac{1}{s^2+1},$$

$$4X(s) - sY(s) - Y(s) = \frac{s}{s^2+1}.$$

Eliminating $Y(s)$ we obtain

$$X(s) = \frac{3s+1}{(s^2+1)(s^2-2s+5)}.$$

The right-hand side can be partial fractioned to the form

$$\frac{As+B}{s^2+1} + \frac{C(s-1)+D}{s^2-2s+5},$$

where the constants A, B, C and D are given by

$$A = 7/10; \quad B = -1/10; \quad C = -7/10; \quad D = 2/5.$$

Thus the solution $x = x(t)$ is given by

$$x(t) = 7/10 \cos t - 1/10 \sin t - 7/10 e^t \cos 2t + 2/5 e^t \sin 2t.$$

Computing the derivative of x and substituting into the first equation yields

$$y(t) = -11/10 e^t \cos 2t - 3/10 e^t \sin 2t + 11/10 \cos t + 7/10 \sin t.$$

Note that in this case we can substitute the function x in the first equation because it is continuously differentiable.

(c) The Laplace transforms of the two equations read

$$sX(s) + Y(s) - X(s) = 0, \tag{1}$$

$$2sX(s) + s^2Y(s) - s + 1 = \frac{e^{-3s}}{s}.$$

Eliminating $Y(s)$ we obtain

$$X(s) = -\frac{e^{-3s}}{s^2(s^2-s-2)} - \frac{s-1}{(s^2-s-2)s}. \tag{2}$$

The partial fractions for the first term read

$$\frac{1}{s^2(s^2-s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s-2}\,,$$

and the system for determining A, B, C, D is

$$\begin{aligned} A + C + D &= 0\,, \\ -A + B - 2C + D &= 0\,, \\ -2A - B &= 0\,, \\ -2B &= 1\,. \end{aligned}$$

The solution to this system is given by $A = 1/4, B = -1/2, C = -1/3, D = 1/12$. The partial fractions corresponding to the second term are given by

$$\frac{s-1}{s(s^2-s-2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2}\,.$$

Here the system for A, B, C is

$$\begin{aligned} A + B + C &= 0\,, \\ -A - 2B + C &= 1\,, \\ -2A &= -1\,, \end{aligned}$$

and the solution to it is $A = 1/2, B = -2/3, C = 1/6$. Taking the inverse transform of (2) we finally obtain

$$x(t) = -1/2 - 1/6e^{2t} + 2/3e^{-t} + (-1/4 + 1/2(t-3) + 1/3e^{-t+3} - 1/12e^{2t-6})u(t-3)\,.$$

Substituting (2) into (1) and inverting similarly the Laplace transform on $Y(s)$ we obtain

$$y(t) = -1/2 + 1/6e^{2t} + 4/3e^{-t} + (-3/4 + 1/2(t-3) + 2/3e^{-t+3} + 1/12e^{2t-6})u(t-3)\,.$$

5. (a) Write $y = y(x)$ as a power series about $x_0 = 0$, i.e., let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and note that $y(0) = a_0 = 0$, $y'(0) = a_1 = 0$. From

$$y'' + 2y' + y = \sum_{n=0}^{\infty} (a_{n+2}(n+2)(n+1) + 2a_{n+1}(n+1) + a_n)x^n = x^2\,,$$

it is clear that y is a solution of the IVP if and only if

$$a_{n+2}(n+2)(n+1) + 2a_{n+1}(n+1) + a_n = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{if } n = 0, 1, \text{ or } n \geq 3. \end{cases}$$

Recall that $a_0 = a_1 = 0$. Hence $a_2 = 0$, $a_3 = 0$, $a_4 = \dfrac{1}{12}$, and

$$a_n = -\frac{2}{n}a_{n-1} - \frac{1}{n(n-1)}a_{n-2} \quad \text{for all } n \geq 5\,.$$

Specifically, $a_5 = -\dfrac{1}{30}$, $a_6 = \dfrac{1}{120}$, $a_7 = -\dfrac{1}{630}$, etc., and so

$$y(x) = \frac{1}{12}x^4 - \frac{1}{30}x^5 + \frac{1}{120}x^6 - \frac{1}{630}x^7 + \dots\,.$$

This power series converges for all real numbers x.

(b) The characteristic equation $r^2 + 2r + 1 = (r+1)^2 = 0$ associated with the homogeneous ODE $y'' + 2y' + y = 0$ has $r = -1$ as a double root, whereas $\alpha + \imath\beta = 0$ is not a root. A correct "trial" solution for the inhomogeneous equation therefore is $y_P(x) = A_0 + A_1 x + A_2 x^2$, and

$$y_P'' + 2y_P' + y_P = (A_0 + 2A_1 + 2A_2) + (A_1 + 4A_2)x + A_2 x^2 = x^2$$

shows that

$$A_0 + 2A_1 + 2A_2 = 0\,, \quad A_1 + 4A_2 = 0\,, \quad A_2 = 1\,,$$

and consequently, $A_0 = 6$, $A_1 = -4$, and $A_2 = 1$. Hence the general solution of $y'' + 2y' + y = x^2$ is given by

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + 6 - 4x + x^2\,,$$

and imposing the initial conditions

$$y(0) = C_1 + 6 = 0\,, \quad y'(0) = -C_1 + C_2 - 4 = 0\,,$$

yields $C_1 = -6$, $C_2 = -2$, and so

$$\begin{aligned} y(x) &= -6e^{-x} - 2xe^{-x} + 6 - 4x + x^2 \\ &= -6\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} - 2\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!} + 6 - 4x + x^2 \\ &= \sum_{n=3}^{\infty} 2(-1)^n \frac{n-3}{n!} x^n = \frac{1}{12}x^4 - \frac{1}{30}x^5 + \frac{1}{120}x^6 - \frac{1}{630}x^7 + \dots\,, \end{aligned}$$

which not only is in perfect agreement with the power series found in (a), but also shows that in fact $a_n = 2(-1)^n(n-3)/n!$ for all $n \geq 3$.

6. Assume

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute into the ODE,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Let $k = n-2$, $k = n$, $k = n-1$, and $k = n$, respectively, in these summations. We obtain

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} k a_k x^k - 2\sum_{k=0}^{\infty} a_{k+1}(k+1)x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

so that

$$2a_2 - 2a_1 - a_0 + \sum_{k=1}^{\infty} \big((k+2)(k+1)a_{k+2} - 2(k+1)a_{k+1} + (k-1)a_k\big)x^k = 0.$$

Hence $a_2 = a_1 + \frac{1}{2}a_0$ where a_0 and a_1 are arbitrary. The recursion formula is

$$a_{k+2} = \frac{2a_{k+1}}{k+2} - \frac{(k-1)a_k}{(k+2)(k+1)}, \quad k = 1, 2, \ldots$$

And the first few coefficients are

$$a_3 = \frac{2}{3}a_1 + \frac{1}{3}a_0, \quad a_4 = \frac{1}{4}a_1 + \frac{1}{8}a_0.$$

Therefore the general solution is

$$y = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \cdots\right) + a_1\left(x + x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \cdots\right).$$

Since $y(0) = -1$ and $y'(0) = 0$, $a_0 = -1$ and $a_1 = 0$, and so the solution to the IVP is

$$y = -1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \cdots.$$

7. Since $p(x) = \dfrac{-x}{1+x^3}$ and $q(x) = \dfrac{3x^2}{1+x^3}$, the singular points occur when

$$1 + x^3 = (1+x)(x^2 - x + 1) = 0,$$

i.e., $x = -1,\ \dfrac{1 \pm i\sqrt{3}}{2}$. Then the distance between $x_0 = 1$ and $x = -1$ is 2, and the distance between x_0 and $x = \dfrac{1 \pm i\sqrt{3}}{2}$ is

$$\sqrt{\left(\frac{1}{2}\right)^2 + \frac{3}{4}} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 < 2.$$

Therefore a lower bound for the radius of convergence of the solution is 1.

8. Assume

$$y = \sum_{n=0}^{\infty} a_n t^n, \quad y' = \sum_{n=1}^{\infty} n a_n t^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}.$$

Then

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^n + \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \sum_{n=0}^{\infty} a_n t^n = 0.$$

Writing out the first few terms,

$$\begin{aligned}(2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 \cdots) + (a_1 t + 2a_t^2 + 3a_3 t^3 + \cdots)& \\ +a_0 + a_0 t + a_0 \frac{t^2}{2!} + a_0 \frac{t^3}{3!} + a_1 t + a_1 t^2 + a_1 \frac{t^3}{2!}& \\ +a_2 t^2 + a_2 t^3 + a_2 \frac{t^4}{2!} + \cdots + a_3 t^3 + \ldots &= 0\,.\end{aligned}$$

Equating the coefficients of the like terms,

$$\begin{aligned}\text{consts.:}\quad 2a_2 + a_0 &= 0 \\ t: \quad 6a_3 + a_1 + a_0 + a_1 &= 0 \\ t^2: \quad 12a_4 + 2a_2 + \frac{a_0}{2} + a_1 + a_2 &= 0 \\ t^3: \quad 20a_5 + 3a_3 + \frac{a_0}{6} + \frac{a_1}{2} + a_2 + a_3 &= 0 \\ \text{etc.}\end{aligned}$$

Since $a_0 = 0$ and $a_1 = -1$, we have

$$a_2 = 0, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{12}, \quad a_5 = -\frac{1}{24},$$

and the solution is

$$\begin{aligned} y(t) &= a_0 + a_1 t + a_2 t^2 + \dots \\ &= -t + \frac{1}{3}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5 + \dots . \end{aligned}$$