Math 334—Solutions to Assignment 3

1. Using the definition determine the Laplace transform of the function:

$$f(t) = \begin{cases} 1-t, & 0 < t < 1, \\ 0, & 1 < t. \end{cases}$$

Solution:

$$\mathcal{L}[f(t)](s) = \int\limits_0^1 e^{-st}(1-t)dt$$

and take note here of the upper limit of integration. Then integration by part yields (for non-zero s)

$$\begin{aligned} \mathcal{L}[f(t)](s) &= \left[\frac{(t-1)}{s}e^{-st} + \frac{1}{s}\int e^{-st}dt\right]_0^1 \\ &= \left[\frac{(t-1)}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^1 \\ &= \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right) \end{aligned}$$

Use the Laplace transform table to determine the following transforms:

2. $\mathcal{L}\{t^4 - t^2 - t + \sin(\sqrt{2}t)\}$.

   Solution:

   Again take $s > 0$.

$$\begin{aligned} \mathcal{L}[t^4 - t^2 - t + \sin\sqrt{2}t](s) &= \mathcal{L}[t^4](s) - \mathcal{L}[t^2](s) - \mathcal{L}[t](s) + \mathcal{L}[\sin\sqrt{2}t \\ &= \frac{24}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}. \end{aligned}$$

3. $\mathcal{L}\{t\sin^2 t\}$

   Solution:

   Recall the trig identity $\sin^2 t = \frac{1}{2}(1-\cos 2t)$. Then

$$\begin{aligned}\mathcal{L}[t\sin^2 t](s) &= \frac{1}{2}\mathcal{L}[t(1-\cos 2t)](s) \\ &= \frac{1}{2}\left[\mathcal{L}[t](s) - \mathcal{L}[t\cos 2t](s)\right] \\ &= \frac{1}{2}\left[\frac{1}{s^2} - \frac{(s^2-4)}{(s^2+4)^2}\right]\end{aligned}$$

   This can be simplified to:

$$2\frac{(3s^2+4)}{s^2(s^2+4)^2}$$

4. Starting with the transform $\mathcal{L}\{1\}(s) = 1/s$, use the formula for the derivative of the Laplace transform to show that $\mathcal{L}\{t^n\} = n!/s^{n+1}$.

   Solution:

   By setting $f(t) = 1$ in the formula for the derivative of a Laplace transform and noting that the Laplace transform of 1 is $1/s$, we get

$$\begin{aligned}\mathcal{L}[t^n](s) &= (-1)^n \frac{d^n}{ds^n}\frac{1}{s} \\ &= (-1)^n(-1)^n\frac{n!}{s^{n+1}}\end{aligned}$$

   by the standard formula for derivatives of $1/s$. But the two factors of $(-1)^n$ combine to give a single factor of $(-1)^{2n}$, which of course equals $+1$, completing the derivation.

Determine the inverse Laplace transform of the functions:

5.

$$\frac{3}{(2s+5)^3}$$

Solution:

$$\frac{3}{(2s+5)^3} = \frac{3}{16}\frac{2}{(s+5/2)^3}$$

then

$$\mathcal{L}^{-1}\left\{\frac{3}{(2s+5)^3}\right\} = 3/16e^{-5/2t}t^2$$

6.

$$\frac{7s^3-2s^2-3s+6}{(s-2)s^3}$$

Solution:

$$\begin{aligned}
\frac{7s^3-2s^2-3s+6}{s^3(s-2)} &= \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-2} \\
7s^3-2s^2-3s+6 &= As^2(s-2)+Bs(s-2)+C(s-2)+Ds^3 \\
&= (A+D)s^3+(-2A+B)s^2+(-2B+C)s-2C \\
A+D=7, \quad -2A+B=-2, &\quad -2B+C=-3, \quad -2C=6 \\
A=1, \quad B=0, &\quad C=-3, \quad D=6, \\
\mathcal{L}\Big\{\frac{7s^3-2s^2-3s+6}{s^2(s-2)}\Big\} &= \mathcal{L}\Big\{\frac{1}{s}\Big\}-3\mathcal{L}\Big\{\frac{1}{s^3}\Big\}+6\mathcal{L}\Big\{\frac{1}{s-2}\Big\} \\
&= 1-\frac{3}{2}t^2+6e^{2t}.
\end{aligned}$$

Solve the given initial value problem using Laplace transforms:

7. $y''+y=t, \quad y(\pi)=0, \quad y'(\pi)=0.$

Solution:
We first change the variable in order to impose the initial conditions at 0. The transform is: $x=t-\pi$. Then the transformed equation and initial values read

$$y''+y=x+\pi \quad y(0)=0, y'(0)=0$$

Applying the Laplace transfrom to both sides yieds

$$s^2Y(s)+Y(s)=\frac{1}{s^2}+\frac{\pi}{s}$$

or

$$Y(s) = \frac{1}{s^2(s^2+1)} + \frac{\pi}{s(s^2+1)}.$$

The partial fractioning of the right hand side gives

$$Y(s) = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{\pi}{s} - \frac{\pi s}{s^2+1}.$$

The inverse Laplace transform then gives

$$y(x) = x - \sin x + \pi - \pi \cos x.$$

After substituting back $x = t - \pi$ and using the trigonometric identities: $\sin(t-\pi) = -\sin t$ and $\cos(t-\pi) = -\cos t$ we finally get

$$y(t) = t + \sin t + \pi \cos t$$

.

8. $y'' + ty' - y = 0, \quad y(0) = 0, \quad y'(0) = 3.$

   Solution:
   Taking the Laplace transform of both sides and taking into account that

   $$\mathcal{L}\{ty'\} = -sY'(s) - Y(s)$$

   we obtain after some re-arrangement

   $$Y' + \left(\frac{2}{s} - s\right) Y = -\frac{3}{s}.$$

   This is a first order linear equation with an integrating factor:

   $$\mu(s) = \exp\left[\int \left(\frac{2}{s} - s\right) ds\right] = s^2 e^{-s^2/2}.$$

Thus

$$Y(s) = \frac{3}{s^2 e^{-s^2/2}} \int s e^{s^2/2} ds = \frac{3}{s^2}\left(1 + Ce^{s^2/2}\right).$$

Since the Laplace transform of any function must tend to zero if s approaches infinity then the solution above is a legitimate Laplace transform only if $C = 0$. Then we can invert the transform and obtain that $y(t) = 3t$.

9. Determine the inverse Laplace transform of: $\dfrac{e^{-s}}{s^2+4}$.

Solution:

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2+4}\right\} = \frac{1}{2}\left[\sin 2(t-1)\right]u(t-1)$$

10. Solve for the current $I(t)$ governed by the initial-value problem: $I'' + 4I = g(t), \quad I(0) = 1, \quad I'(0) = 3$ where

$$g(t) = \begin{cases} 3\sin t, & 0 < t < 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

First write g(t) in terms of unit step functions.

Solution:
The function $g(t)$ can be represented as $g(t) = 3\sin t - 3\sin t u(t-2\pi)$. The Laplace transform of the equation then becomes:

$$s^2J(s) - s - 3 + 4J(s) = \frac{3}{s^2+1} - e^{-2\pi s}\frac{3}{s^2+1}$$

where $J(s)$ is the Laplace transform of $I(t)$. Then

$$J(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4} + \frac{3}{(s^2+1)(s^2+4)} - e^{-2\pi s}\frac{3}{(s^2+1)(s^2+4)}$$

After computing the partial fractions corresponding to

$$\frac{3}{(s^2+1)(s^2+4)}$$

and inverting the Laplace transform we obtain

$$I(t) = \sin t + \sin 2t + \cos 2t + (\frac{1}{2}\sin 2t - \sin t)u(t-2\pi).$$

11. Solve the initial-value problem: $y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi), \quad y(0) = 0, \quad y'(0) = 0.$

Solution:
Take the Laplace transform of the DE to get:

$$\begin{aligned} s^2Y(s) + 4sY(s) + 4Y(s) &= \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s} \\ \Rightarrow \quad Y(s) &= \frac{1}{s(s+2)^2}\left(e^{-\pi s} - e^{-2\pi s}\right) \end{aligned}$$

Using the partial fractions decomposition for a denominator containing a repeated linear factor

$$\frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

we find $A = 1/4$, $B = -1/4$, $C = -1/2$, so

$$\begin{aligned} Y(s) &= \left(\frac{1}{4s} - \frac{1}{4(s+2)} - \frac{1}{2(s+2)^2}\right)e^{-\pi s} \\ &\quad - \left(\frac{1}{4s} - \frac{1}{4(s+2)} - \frac{1}{2(s+2)^2}\right)e^{-2\pi s} \end{aligned}$$

The inverse transform results in

$$\begin{aligned} y(t) &= \frac{1}{4}u(t-\pi)\left(1 - e^{-2(t-\pi)} - 2(t-\pi)e^{-2(t-\pi)}\right) \\ &\quad - \frac{1}{4}u(t-2\pi)\left(1 - e^{-2(t-2\pi)} - 2(t-2\pi)e^{-2(t-2\pi)}\right) \end{aligned}$$