Math 334— Solutions to Assignment 2

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1. Find a particular solution to each of the following equations for $y = y(t)$:

   (a) $y'' + 2y' + y = 10 + 2e^{-t}$

   Solution:

   According to the principle of superposition, we seek the solution as $y_p = y_{p,1} + y_{p,2}$ where $y_{p,1}$ solve the equation with a right hand side 10 and $y_{p,2}$ solve it with the right hand side $2e^{-t}$.

   It is straightforward to determine that $y_{p,1} = 10$. Since the characteristic equation has a repeated root of -1, the guess for $y_{p,2}$ is $y_{p,2} = At^2e^{-t}$. The first two derivatives of this guess are: $y'_{p,2} = 2Ate^{-t} - At^2e^{-t}$ and $y''_{p,2} = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$, and after we plug the in the left had side of the equation, together with $y_{p,2}$, we obtain that $2Ae^{-t} = 2e^{-t}$ i.e. $A = 1$. Therefore one particular solution to the original equation is given by

   $$\boxed{y_p(t) = 10 + t^2e^{-t}.}$$

   (b) $y'' + 2y' + 2y = 4te^{-t}\cos t$.

   Solution:

   The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $-1 \pm i$. The non-homogenous term is $f(t) = 4te^{-t}\cos t$, so we try a solution of the form: $y(t) = t(A_1t + A_0)e^{-t}\sin t + t(B_1t + B_0)e^{-t}\cos t$. Note that we include the factor t because $r = -1 + i$ is a root of the characteristic polynomial. We then compute

   $$\begin{aligned} y' = &\quad [(A_1 - B_1)t^2 + (2B_1 + A_0 - B_0)t + B_0]e^{-t}\cos t \\ &-[(A_1 + B_1)t^2 + (-2A_1 + A_0 + B0)t - A_0]e^{-t}\sin t \\ y'' = &-2[A_1t^2 + (2B_1 - 2A_1 + A_0)t + B_0 - A_0 - B_1]e^{-t}\cos t \\ &-2[-B_1t^2 + (2A_1 + 2B_1 - B_0)t + A_0 + B_0 - A_1]e^{-t}\sin t \end{aligned}$$

   and plug into our equation:

   $$y'' + 2y' + 2y = 2(2tA_1 + B_1 + A_0)e^{-t}\cos t - 2(2tB_1 - A_1 + B_0)e^{-t}\sin t = 4te^{-t}\cos t.$$

   This fixes $A_0 = B_1 = 0$ and $A_1 = B_0 = 1$. Thus, a particular solution to this ODE is

   $$\boxed{y(t) = te^{-t}(\cos t + t\sin t).}$$

   Note, that the work can be simplified if we notice that the guess for the particular solution can be written as: $y(t) = p_1y_1 + p_2y_2 = t(A_1t + A_0)e^{-t}\sin t + t(B_1t + B_0)e^{-t}\cos t$ where p_1 and p_2 are the polynomials multiplying the solutions to the homogeneous equation: $y_1 = e^{-t}\sin t, y_2 = e^{-t}\cos t$. Then, we can differentiate y in this abstract form, and use the fact that y_1, y_2 are solutions to the homogeneous equation that will simplify significantly the computations.

(c) $y'' + 2y' + y = 3e^{-t}\sqrt{t+1}$.

Solution:

Consider the associated homogeneous equation

$$y'' + 2y' + y = 0.$$

The auxiliary equation is $r^2 + 2r + 1 = 0$. It has one double (real) root $r = -1$. Therefore $y_1 = e^{-t}$ and $y_2 = te^{-t}$ are linearly independent solutions of the homogeneous equation. We will use the method of variation of parameters to find a particular solution of the nonhomogeneous equation of the form

$$y = v_1(t)e^{-t} + v_2(t)te^{-t}.$$

The functions v_1 and v_2 satisfy the system of equations

$$\begin{cases} v_1' e^{-t} + v_2' te^{-t} = 0 \\ -v_1' e^{-t} + v_2'(e^{-t} - te^{-t}) = 3e^{-t}\sqrt{t+1} \end{cases}$$

Adding these two equations, we get

$$v_2' e^{-t} = 3e^{-t}\sqrt{t+1}$$
$$v_2' = 3\sqrt{t+1}$$

So,

$$v_2 = 2(t+1)^{3/2}.$$

Since $v_2' = 3\sqrt{t+1}$, from the first equation of the system we find

$$v_1' = -v_2' t = -3t\sqrt{t+1}$$

$$v_1 = -3\int t\sqrt{t+1}dt$$

Substitution: $z = t+1$, $dz = dt$.

$$v_1 = -3\int (z-1)\sqrt{z}dz = -3\int (z^{3/2} - z^{1/2})dz = -3(\frac{2}{5}z^{5/2} - \frac{2}{3}z^{3/2}) = -\frac{6}{5}z^{5/2} + 2z^{3/2}$$
$$= -\frac{6}{5}(t+1)^{5/2} + 2(t+1)^{3/2}.$$

We did not add a constant of integration since we are looking for particular functions v_1 and v_2.

So, a particular solution of the original equation is

$$\begin{aligned} y &= \left(-\frac{6}{5}(t+1)^{5/2} + 2(t+1)^{3/2}\right)e^{-t} + 2(t+1)^{3/2}te^{-t} \\ &= \left(-\frac{6}{5}(t+1)^{5/2} + 2(t+1)^{3/2} + 2t(t+1)^{3/2}\right)e^{-t} \\ &= \left(-\frac{6}{5}(t+1)^{5/2} + (2+2t)(t+1)^{3/2}\right)e^{-t} \\ &= \left(-\frac{6}{5}(t+1)^{5/2} + 2(t+1)^{5/2}\right)e^{-t} \\ &= \boxed{\frac{4}{5}(t+1)^{5/2}e^{-t}} \end{aligned}$$

(d) $y'' + 3y' + 2y = \frac{1}{e^t+1}$.

Solution:

Consider the associated homogeneous equation $y'' + 3y' + 2y = 0$. The auxiliary equation is $r^2 + 3r + 2 = 0$. It has two real roots $r_1 = -1$, $r_2 = -2$. Therefore $y_1 = e^{-t}$ and $y_2 = e^{-2t}$ are linearly independent solutions of the homogeneous equation. We will use the method of variation of parameters to find a particular solution of the nonhomogeneous equation of the form

$$y = v_1(t)e^{-t} + v_2(t)e^{-2t}.$$

The functions v_1 and v_2 satisfy the system of equations

$$\begin{cases} v_1'e^{-t} + v_2'e^{-2t} = 0 \\ -v_1'e^{-t} - v_2'2e^{-2t} = \frac{1}{e^t+1} \end{cases}$$

Adding these two equations, we get

$$-v_2'e^{-2t} = \frac{1}{e^t+1}$$

$$v_2' = -\frac{e^{2t}}{e^t+1}$$

$$v_2 = -\int \frac{e^{2t}}{e^t+1}dt$$

Substitution:$z = e^t$, $dz = e^t dt$

$$v_2 = -\int \frac{z}{z+1}dz = -\int \frac{z+1-1}{z+1}dz = \int \left(-1 + \frac{1}{z+1}\right) dz = -z + \ln|z+1|$$

$$= -e^t + \ln(e^t+1)$$

Using the first equation of the system and the fact that $v_2' = -\frac{e^{2t}}{e^t+1}$, we get

$$v_1' = \frac{e^t}{e^t+1}$$

$$v_1 = \int \frac{e^t}{e^t+1}dt$$

Substitution:$z = e^t$, $dz = e^t dt$

$$v_1 = \int \frac{1}{z+1}dz = \ln|z+1| = \ln(e^t+1)$$

So, we get the following particular solution of the original differential equation:

$$y = \ln(e^t+1)e^{-t} + (-e^t + \ln(e^t+1))e^{-2t} = \ln(e^t+1)e^{-t} - e^{-t} + \ln(e^t+1)e^{-2t}$$

Since e^{-t} is a solution of the associated homogeneous solution, we may disregard this term. So, a particular solution of the original equation is

$$\boxed{y = \left(e^{-t} + e^{-2t}\right)\ln(e^t+1).}$$

2. Find the general solution to the equation for $y = y(x)$,

$$y'' + 4y' + 4y = e^{-2x} \sec^2 x,$$

   (a) by using reduction of order (*what would be a good choice for a solution to the homogeneous equation?*); and
   (b) by using variation of parameters.

Solution:

   (a) We observe that $y_1 = e^{-2x}$ is a solution to the associated homogeneous equation. Then we try a solution of the form $y = ue^{-2x}$ and obtain: $y' = e^{-2x}(u' - 2u)$ and $y'' = e^{-2x}(u'' - 4u' + 4u)$. Subbing into the differential equation and simplifying we obtain: $u'' = \sec^2 x$. Integrating this equation twice yields: $u = -\ln|\cos x| + c_1 x + c_2$. Therefore the general solution to the equation is:

$$\boxed{y(x) = e^{-2x}(-\ln|\cos x| + c_1 x + c_2).}$$

   (b) The characteristic polynomial of the equation is $r^2 + 4r + 4 = (r+2)^2$ which has $r = -2$ as a double root, so that two linearly independent solutions of the homogeneous equation associated with the original equation are given by: $y_1 = e^{-2x}, y_2 = xe^{-2x}$. Therefore, we should guess a particular solution for the method of variation of parameters of the form: $v_1 y_1 + v_2 y_2$. Then $v_1' y_1 + v_2' y_2 = 0$ and $v_1' y_1' + v_2' y_2' = e^{-2x} \sec^2 x$. Substituting y_1, y_2 gives the linear system for v_1', v_2':

$$\begin{aligned} v_1' + v_2' x &= 0 \\ -2v_1' + v_2'(1 - 2x) &= sec^2 x. \end{aligned}$$

   Its solution is given by $v_1' = x \sec^2 x, v_2' = -\sec^2 x$. Integrating once by parts we obtain $v_1 = x \tan x + \ln|\cos x|$. For v_2 we obtain directly $v_2 = -\tan x$. Therefore, the general solution to the equation is:

$$\boxed{y(x) = e^{-2x}(-\ln|\cos x| + c_1 x + c_2).}$$

3. Find general solutions to the equations for $y = y(t)$:

   (a)

$$4t^2 y'' - 8ty' + 9y = 0, \quad t > 0;$$

   Solution:
   We transform the variable t as $t = e^x$ or $x = \ln t$ that turns the equation into a constant coefficients equation: $4y'' - 12y' + 9y = 0$. The root of its characteristic equation has a repeated root $3/2$. Then the solution is given by $y(x) = c_1 e^{3/2x} + c_2 x e^{3/2x}$. Plugging in $x = \ln t$ yields: $\boxed{y = (c_1 + c_2 \ln t)t^{3/2}.}$
   Alternatively, we can seek a solution of the form $y = t^r$. Then $y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$.

$$4t^2 r(r-1)t^{r-2} - 8trt^{r-1} + 9t^r = 0.$$

$$4r(r-1)t^r - 8rt^r + 9t^r = 0.$$

$$4r(r-1) - 8r + 9 = 0.$$
$$4r^2 - 12r + 9 = 0.$$
$$r = \frac{12 \pm \sqrt{144 - 144}}{8} = \frac{3}{2}.$$

One nontrivial solution is $t^{3/2}$. Since $3/2$ is a double root of the above characteristic equation, another nontrivial solution is $t^{3/2} \ln t$.
So, the general solution of the differential equation is

$$\boxed{y = (c_1 + c_2 \ln t)t^{3/2}.}$$

(b)

$$2t^2y'' + 4ty' - 2y = 0, \quad t < 0.$$

Solution:
Again we transform t to $t = -e^x$ (note that this time the domain of the solution is $t < 0$). This produces the equation $2y(x)'' + 2y'(x) - 2y(x) = 0$, that has a general solution

$$y(x) = c_1 e^{\frac{-1+\sqrt{1+4}}{2}x} + c_2 e^{\frac{-1-\sqrt{1+4}}{2}x}.$$

Plugging in $x = ln(-t)$ we obtain the general solution for $t < 0$ as

$$\boxed{y = c_1(-t)^{\frac{-1+\sqrt{5}}{2}} + c_2(-t)^{\frac{-1-\sqrt{5}}{2}}.}$$

Alternatively, we can seek a solution of the form $y = (-t)^r$. Then $y' = -r(-t)^{r-1}$, $y'' = r(r-1)(-t)^{r-2}$.

$$2t^2r(r-1)(-t)^{r-2} - 4tr(-t)^{r-1} - 2(-t)^r = 0$$
$$2r(r-1)(-t)^r + 4r(-t)^r - 2(-t)^r = 0$$
$$2r(r-1) + 4r - 2 = 0$$
$$2r^2 + 2r - 2 = 0$$
$$r^2 + r - 1 = 0$$
$$r_1 = \frac{-1+\sqrt{5}}{2}, \quad r_2 = \frac{-1-\sqrt{5}}{2}.$$

We get two linearly independent solutions $(-t)^{\frac{-1+\sqrt{5}}{2}}$ and $(-t)^{\frac{-1-\sqrt{5}}{2}}$.
So, the general solution of the differential equation for $t < 0$ is

$$\boxed{y = c_1(-t)^{\frac{-1+\sqrt{5}}{2}} + c_2(-t)^{\frac{-1-\sqrt{5}}{2}}.}$$