**Math 334— Solutions to Assignment 1**

1. Solve the initial value problem:

$$x^2dx + 2ydy = 0, \quad y(0) = 2.$$

Solution:

$$\begin{aligned} & x^2dx + 2ydy = 0 \quad , \quad y(0) = 2 \\ \Rightarrow \quad & \int 2ydy = -\int x^2dx \\ \Rightarrow \quad & y^2 = -\frac{1}{3}x^3 + C \end{aligned}$$

Applying the initial condition gives us that $y^2(0) = C = 4$. Then

$$y^2 = 4 - \frac{1}{3}x^3 \quad .$$

Note that this solution is not defined for some values of $x$ for which $4 - \frac{1}{3}x^3 < 0$.

2. Find the general solution to the equation:

$$x\frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x}.$$

Solution:

The integrating factor is given by:

$$\mu'(x) = \frac{3\mu}{x}$$

and after integration we find that $\mu(x) = x^3$. Multiply the standard form of the equation by $\mu$ to obtain

$$(x^3y)' = -3x^4 + x\sin x$$

Integrating both sides we get

$$x^3y = -3/5x^5 + \int x\sin x dx.$$

integrating the integral above by parts and dividing the equation by $x^3$ we obtain

$$y = -3/5x^2 - \cos x/x^2 + \sin x/x^3 + C/x^3.$$

3. Solve the initial value problems:

$$(ye^{xy} - 1/y)dx + (xe^{xy} + x/y^2)dy = 0, \quad y(1) = 1,$$

Solution:

Using the test for exactness one verifies that the equation is exact. Then

$$\partial F/\partial x = ye^{xy} - 1/y$$

and after integration

$$F(x, y) = e^{xy} - x/y + g(y) \tag{1}$$

Differentiating with respect to $y$ we get

$$\partial F/\partial y = xe^{xy} + x/y^2 + g'(y)$$

On the other hand we need that

$$\partial F/\partial y = xe^{xy} + x/y^2$$

and therefore $g'(y) = 0$. Thus we can choose $g(y) = 0$ because any constant can be absorbed in the right hand side constant of the solution which we easily obtain from (1) to be:

$$e^{xy} - \frac{x}{y} = const = C$$

Since we want $y(1) = 1$ it follows that

$$F(1, 1) = e - 1 = C$$

and then the solution of the initial-value problem (IVP) is given by

$$e^{xy} - \frac{x}{y} = e - 1.$$

$$(x+2)\sin y + (x\cos y)y' = 0, \quad y(1) = \pi/2.$$

Solution: $(x+2)\sin y + (x\cos y)y' = 0 \Leftrightarrow (x+2)\sin y\mathrm{d}x + (x\cos y)\mathrm{d}y = 0$.

$$M_y = (x+2)\cos y \neq N_x = \cos y.$$

Separation of variables
or
Find an integrating factor $\mu(x) = xe^x$ $\left(\frac{M_y - N_x}{N} = \frac{x+1}{x} \Rightarrow \frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{x+1}{x}\mu\right)$, then $(x^2+2x)e^x \sin y\mathrm{d}x + x^2e^x\cos y\mathrm{d}y = 0$ is exact.
Follow the standard process $\Rightarrow x^2e^x\sin y = C$.
Apply the initial condition to obtain $C = e$, then the particular solution to IVP is $x^2e^x\sin y = e$.

4. Find the general solution to the equations:

$$\frac{dy}{dx} - y = e^{2x}y^3$$

Solution: We first divide by $y^3$ and obtain

$$y^{-3}\frac{dy}{dx} - y^{-2} = e^{2x}.$$

Note that while dividing by $y$ we presume that $y \neq 0$ but $y = 0$ is a solution to the initial equation which will not be a solution to the new equation. Therefore, we have to keep it in mind. We further substitute $y^{-2} = u$ and after some simple calculus $(du/dx = -2y^{-3}dy/dx)$ obtain

$$\frac{du}{dx} + 2u = -2e^{2x}.$$

This is a linear equation and the integrating factor is $\mu(x) = e^{2x}$. Multiplying the equation with it we obtain

$$\frac{d}{dx}(e^{2x}u) = -2e^{4x}$$

and integrating once, and substituting back $u = y^{-2}$

$$y^{-2} = -\frac{1}{2}e^{2x} + ce^{-2x}.$$

So the solutions are $y^{-2} = -\frac{1}{2}e^{2x} + ce^{-2x}$ and $y = 0$.

$$\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}.$$

Solution:

Note that x and y must be both positive for the logarithms to be well defined.

The equation can be put in the form

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{y}{x} + 1\right).$$

This is a homogeneous equation and we need to substitute $u = y/x$. Then the equation in terms of $u$ reads

$$x\frac{du}{dx} + u = u(\ln u + 1).$$

This is a separable equation and the separated form is

$$\frac{du}{u\ln u} = \frac{dx}{x}.$$

After one integration we obtain

$$\ln\ln|u| = \ln|x| + c_1$$

and taking the exponents of both sides

$$\ln|u| = cx.$$

This yields

$$y = xe^{cx}.$$

$$\frac{dy}{d\theta} + \frac{y}{\theta} = -4\theta y^{-2},$$

Solution:

The equation is of a Bernoulli type with $n = -2$. Therefore we substitute $v = y^3$ which yields

$$\frac{dy}{d\theta} = 1/3y^{-2}\frac{dv}{d\theta}$$

The equation transforms into

$$1/3y^{-2}\frac{dv}{d\theta} + \frac{y}{\theta} = -4\theta y^{-2}$$

or (multiplying by $3y^2$ and substituting $y^3 = v$)

$$\frac{dv}{d\theta} + \frac{3}{\theta}v = -12\theta$$

which is a linear equation. $\mu(x) = \theta^3$ and multiplying by it we get

$$\frac{d}{d\theta}[\theta^3 v] = -12\theta^4$$

which after one integration gives

$$v = -12/5\theta^2 + c/\theta^3$$

or

$$y = (-12/5\theta^2 + c/\theta^3)^{1/3}$$

5. Find the general solution to the second order equation:

$$y'' - 5y' + 6y = 0$$

Solution:

The auxiliary equation is $r^2 - 5r + 6 = 0$. Use the quadratic formula to find roots at $r = 2$ and $r = 3$, so

$$y(x) = c_1e^{2x} + c_2e^{3x} \quad .$$

6. Solve the second-order initial value problem:

$$y'' - 6y' + 9y = 0; \quad y(0) = 2, y'(0) = 25/3.$$

Solution:

The auxiliary equation has a repeated root at $r = 3$, so the general solution is

$$y(x) = (c_1 + c_2 x)\, e^{3x} \quad .$$

The condition $y(0) = 2$ fixes $c_1 = 2$. Now

$$y'(x) = (3c_1 + c_2 + 3c_2 x)\, e^{3x} \quad .$$

so

$$y'(0) = 3c_1 + c_2 = 6 + c_2 \quad .$$

The condition $y'(0) = 25/3$ fixes $c_2 = 7/3$ so the particular solution is

$$y(x) = \left(2 + \frac{7}{3}x\right) e^{3x}.$$

7. To see the effect of changing the parameter b in the initial value problem

$$y'' + by' + 4y = 0; \quad y(0) = 1, y'(0) = 0,$$

solve the problem for b=5, 4, and 2. Graph the solutions on your graphing calculators to observe the change in behaviour. You do not need to present the graphs with the assignment solutions.

The auxiliary equations for $b = 5, 4, 2$ are

$$r^2 + 5r + 4 = 0, \quad r^2 + 4r + 4 = 0, \quad r^2 + 2r + 4 = 0$$

with roots

$$r_1 = -1, r_2 = -4; \quad r_1 = r_2 = -2; \quad r_1 = -1 + \sqrt{(3)}i, r_2 = -1 - \sqrt{(3)}i;$$

This implies the general solutions

$$y(x) = c_1 e^{-x} + c_2 e^{-4x}, \quad y(x) = c_1 e^{-2x} + c_2 x e^{-2x}, \quad y(x) = e^{-x}(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x))$$

Imposing the initial conditions we obtain for the constants $c_1, c_2$ (in each of the cases considered above):

$$c_1 = 4/3, c_2 = -1/3; c_1 = 1, c_2 = 2; c_1 = 1, c_2 = 3^{-1/2}$$