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Aggregation and classification of spatial dynamics of vector-borne disease in advective heterogeneous environment

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Abstract

In this paper, we formulate and analyze a reaction-diffusion-advection vector-borne disease model with spatial heterogeneity. We find the aggregation phenomenon of endemic equilibrium and classify possible dynamics for the model, including the asymptotic profiles and monotonicity of basic reproduction ratio $\mathfrak{R}_0$ with respect to the diffusion and advection rates of infected hosts and vectors. More importantly, we obtain some crucial and interesting phenomena by classifying the level set of $\mathfrak{R}_0$. Specifically, there exist unique critical surfaces to separate the dynamics, namely, the disease-free equilibrium is stable on one side of the surface and unstable on the other side. The resulting aggregation phenomenon shows that the infected individuals will aggregate in the downstream end if their advection rates are sufficiently large relative to dispersal. To the best of our knowledge, the conclusions of the paper complement the results of vector-borne disease in non-advective environments for the first time and provide new perspectives for the investigation and control of the disease.

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1. Introduction

Vector-borne disease is an infectious disease that transmits the virus to hosts through vectors. Some well-known such diseases include mosquito-borne diseases and tick-borne diseases. Mathematical models are powerful tools to study the spread and control of vector-borne disease. There is growing evidence that many factors, such as natural landscape, urbanity, and vector activities, can cause spatial heterogeneity [1,26,31,35]. Accordingly, the heterogeneity of habitat plays a crucial role in vector-borne diseases. So far, most vector-borne disease models adopt bilinear [34,38,43], general [21,33], standard incidence [39] or saturation incidence [9,25]. However, the combination of saturation and standard incidences may be better to give a reasonable qualitative description for the disease when the total host and vector populations and the number of infected hosts and vectors are both large.

As is known to all, hosts and vectors (such as mosquitoes, birds, ticks) in some circumstances may move passively in specific directions owing to the influence of external environments such as water flow and wind, resulting in non-negligible impacts on disease transmission [8,22,24,28]. In general, this process can be described through incorporating the advection term(s) into the model. It seems thus imperative to take advection effect into account in vector-borne disease modeling. To our knowledge, however, very few studies seem to focus on the effect of advective heterogeneous environments on vector-borne diseases.

1.1. Model and basic assumptions

Motivated by the above works, we in current paper consider the following reaction-diffusion-advection vector-borne disease model with spatial heterogeneity:

$$\begin{cases} S_{ht} = D_S S_{hxx} - c_1 S_{hx} + \Lambda(x) - \dfrac{\beta_1(x) S_h f_1(x, I_v)}{S_h + I_h + R_h} - \mu_h(x) S_h, & t > 0,\ x \in (0, L), \\ I_{ht} = D_I I_{hxx} - c_1 I_{hx} + \dfrac{\beta_1(x) S_h f_1(x, I_v)}{S_h + I_h + R_h} - (\mu_h(x) + \gamma_h(x)) I_h, & t > 0,\ x \in (0, L), \\ R_{ht} = D_R R_{hxx} - c_1 R_{hx} + \gamma_h(x) I_h - \mu_h(x) R_h, & t > 0,\ x \in (0, L), \\ S_{vt} = d_S S_{vxx} - c_2 S_{vx} + M(x) - \dfrac{\beta_2(x) S_v f_2(x, I_h)}{S_v + I_v} - \mu_v(x) S_v, & t > 0,\ x \in (0, L), \\ I_{vt} = d_I I_{vxx} - c_2 I_{vx} + \dfrac{\beta_2(x) S_v f_2(x, I_h)}{S_v + I_v} - \mu_v(x) I_v, & t > 0,\ x \in (0, L), \\ D_S S_{hx} - c_1 S_h = D_I I_{hx} - c_1 I_h = D_R R_{hx} - c_1 R_h = 0, & t > 0,\ x = 0, L, \\ d_S S_{vx} - c_2 S_v = d_I I_{vx} - c_2 I_v = 0, & t > 0,\ x = 0, L, \\ S_h(0, x) = \varphi_1(x) \ge 0,\ I_h(0, x) = \varphi_2(x) \ge 0, \not\equiv 0,\ R_h(0, x) = \varphi_3(x) \ge 0, & x \in (0, L), \\ S_v(0, x) = \varphi_4(x) \ge 0,\ I_v(0, x) = \varphi_5(x) \ge 0, \not\equiv 0, & x \in (0, L), \end{cases} \tag{1.1}$$

where u_x and u_{xx} denote the first and second derivatives of u with respect to (w.r.t.) x, respectively, $u \in \{S_h, I_h, R_h, S_v, I_v\}$; $S_h(t,x)$, $I_h(t,x)$ and $R_h(t,x)$ are the spatial densities of susceptible, infected and recovered hosts, and $S_v(t,x)$ and $I_v(t,x)$ are the spatial densities of susceptible and infected vectors at time t and location x in the bounded interval $[0, L]$, respectively; L represents the size of habitat, and $x = 0$ and $x = L$ denote the upstream and downstream end, respectively; The diffusion rates of hosts and vectors are denoted by D_S, D_I, D_R and d_S, d_I, respectively, and are positive; c_1 (c_2) represents the advection rate of hosts (vectors); The external supplies of hosts (vectors) at x are represented by $\Lambda(x)$ ($M(x)$); The terms $\frac{\beta_1(x)S_h}{S_h+I_h+R_h} \cdot f_1(x, I_v)$ and $\frac{\beta_2(x)S_v}{S_v+I_v} \cdot f_2(x, I_h)$ are the force of infection at time t and location x, where $\beta_1(x) = a_1\beta_{11}(x)$, $\beta_2(x) = a_2\beta_{22}(x)$, and a_1 (a_2) is a constant transmission ability per interaction from an infected vector (host); $\beta_{11}(x)$ is the contact rate of a vector at x, and $\beta_{22}(x)$ is the number of vector contact received by a host at x. Note that when the host and vector densities increase, their infectivity may reach saturation. In other words, it may not be practical to take $f_1(x, I_v) \equiv I_v$ and $f_2(x, I_h) \equiv I_h$ since the standard incidences $\frac{\beta_1(x)S_hI_v}{S_h+I_h+R_h}$ and $\frac{\beta_2(x)S_vI_h}{S_v+I_v}$ make sense only when the total host and vector populations are large while the number of infected hosts and vectors is small [16,17]; $\mu_h(x)$ ($\mu_v(x)$) is the death rate of hosts (vectors) at x; The recovery rate of infected hosts is represented by $\gamma_h(x)$ at x; It should be pointed out that c_1 and c_2 should be nonnegative on account of the downstream end defined by $x = L$, and the no-flux boundary conditions indicate that no hosts and vectors pass through the upstream end $x = 0$ and the downstream end $x = L$. Moreover, we assume that other parameters of (1.1) are Hölder continuous functions in $C^{\nu}([0, L])$ with $\nu \in (0, 1)$. For simplicity, let $\alpha_h(\cdot) = \mu_h(\cdot) + \gamma_h(\cdot)$ and

$$g_1(\cdot, S_h, I_h, R_h, I_v) = \beta_1(\cdot)\frac{S_h f_1(\cdot, I_v)}{S_h + I_h + R_h}, \quad g_2(\cdot, S_v, I_v, I_h) = \beta_2(\cdot)\frac{S_v f_2(\cdot, I_h)}{S_v + I_v}.$$

We first introduce some notations. Define

$$\mathfrak{R}_0^{\rm loc}(x) := \sqrt{\mathfrak{R}_0^{vh}(x)\mathfrak{R}_0^{hv}(x)} \text{ and } \mathfrak{R}_{0a}^{\rm loc} := \sqrt{\mathfrak{R}_{0a}^{vh}\mathfrak{R}_{0a}^{hv}},$$

where

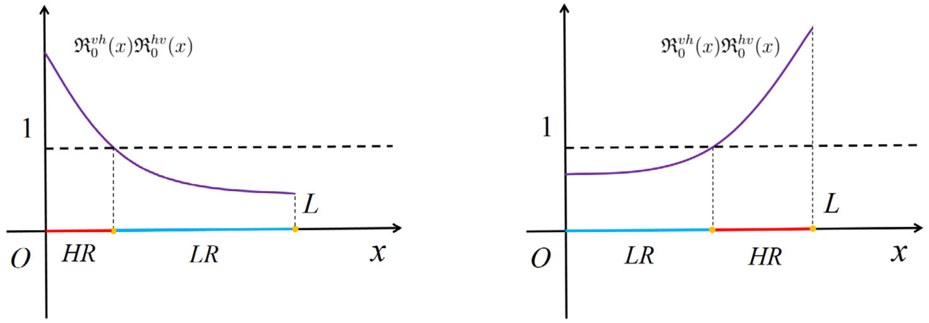
$$\mathfrak{R}_0^{vh}(x) := \frac{\hat\beta_1(x)}{\alpha_h(x)}, \quad \mathfrak{R}_0^{hv}(x) := \frac{\hat\beta_2(x)}{\mu_v(x)} \text{ and } \mathfrak{R}_{0a}^{vh} := \frac{\int_0^L \hat\beta_1(x)\mathrm{d}x}{\int_0^L \alpha_h(x)\mathrm{d}x}, \quad \mathfrak{R}_{0a}^{hv} := \frac{\int_0^L \hat\beta_2(x)\mathrm{d}x}{\int_0^L \mu_v(x)\mathrm{d}x},$$

where $\hat\beta_1(x) = \beta_1(x)\partial_{I_v} f_1(x, 0)$ and $\hat\beta_2(x) = \beta_2(x)\partial_{I_h} f_2(x, 0)$, $x \in (0, L)$. Here, $\mathfrak{R}_0^{\rm loc}(x)$ is called the local basic reproduction ratio of model (1.1) at location x. Biologically, $\mathfrak{R}_0^{vh}(x)$ ($\mathfrak{R}_0^{hv}(x)$) measures the impact of one infected vector (host) on susceptible hosts (vectors) at location x (see, e.g., [23]). Furthermore, the habitat is said to be a high-risk area if $\mathfrak{R}_{0a}^{\rm loc} > 1$, and be a low-risk area if $\mathfrak{R}_{0a}^{\rm loc} < 1$. Define the sets

$$\text{HR} := \{x \in (0, L) : \ \mathfrak{R}_0^{\rm loc}(x) > 1\} \text{ and } \text{LR} := \{x \in (0, L) : \ \mathfrak{R}_0^{\rm loc}(x) < 1\},$$

wherein HR and LR are referred to as high-risk and low-risk sites, respectively.

Throughout this paper, we make the following basic assumptions:



(a) (b)

Fig. 1. Schematic diagram of $\mathfrak{R}_0^{\text{loc}}(x) - 1$ in $x \in (0, L)$. (a) The case (H1); (b) The case (H2).

(A1) $f_i(x, I) \in C^2((0, L) \times \mathbb{R}_+)$ is nonnegative and $f_i(x, I) = 0$ if and only if $I = 0$; $f_i(x, I) < \infty$ as $I \to \infty$; $\partial_I f_i(x, I)$ is positive, and $\partial_I^2 f_i(x, I) \leq 0$ in $(0, L) \times \mathbb{R}_+$, $i = 1, 2$.
(A2) D_I, d_I and c_1, c_2 satisfy $c_1/D_I = c_2/d_I := \sigma$.
(A3) HR and LR are nonempty, and $\mathfrak{R}_0^{\text{loc}}(x) - 1 = 0$ has only one solution for $x \in (0, L)$.

And consider the following two scenarios:

(H1) For any $x_0 \in$ HR and $y_0 \in$ LR, then $x_0 < y_0$ (see Fig. 1 (a)).
(H2) For any $x_0 \in$ HR and $y_0 \in$ LR, then $x_0 > y_0$ (see Fig. 1 (b)).

Remark 1.1. Some frequently used incidence rates fulfill the assumption (A1). For instance, $f_i(x, I) = \frac{I}{1+\rho_i I}$ [25] and $f_i(x, I) = \frac{I}{\rho_i + I}$ [9], $\rho_i > 0$, $i = 1, 2$. Mathematically, the assumption (A2) is a technical condition. In a biological sense, the hypothesis seems to be reasonable which suggests that the movement strategies of hosts and vectors are proportional, i.e., c_1/D_I and c_2/d_I have the same scale. The assumption (A3) means that there is only one high-risk area and one low-risk area in habitat $(0, L)$. From an epidemiological point of view, the scenario (H1) indicates that the upstream and downstream end belong to a high-risk site and a low-risk site, respectively, while the upstream and downstream end belong to a low-risk site and a high-risk site, respectively, as described in scenario (H2).

1.2. Motivation and goal

In epidemiology, the basic reproduction ratio $\mathfrak{R}_0$ is one of the most important concepts which is a crucial threshold for disease outbreak or not [30]. There are a lot of investigations on $\mathfrak{R}_0$, and readers can refer to [2,36,37,45] and references therein. It is known that for the heterogeneous epidemic models, including vector-borne disease models, the basic reproduction ratio is inevitably associated with diffusion and advection rate(s) [1,7,13]. One natural question is thereby how autonomous movement (dispersal) and passive movement (advection) of individuals affect the spread of vector-borne disease in a spatially heterogeneous environment.

Up to now, there are many excellent works on the asymptotic behaviors and monotonicity of $\mathfrak{R}_0$ and steady states w.r.t. diffusion rates [1,4,11,20,23,27,41] and advection rates [5–7]. In [4], Chen and Shi generalized the results of asymptotic behaviors for $\mathfrak{R}_0$ on large and small diffusion to a more general reaction-diffusion compartmental model. A spatial SEIRS model in heterogeneous environment was considered, and the properties of $\mathfrak{R}_0$ were studied by Song et al. [27]. Very recently, Zhao et al. [44] proposed a mosquito-borne disease model with spatial heterogeneity and discussed the monotonicity and asymptotic profiles of $\mathfrak{R}_0$ w.r.t. the diffusion rates of infected individuals. In [5–7], Cui and his collaborators studied a spatial SIS model in advective heterogeneous environments, and investigated the asymptotic behaviors of $\mathfrak{R}_0$, and the concentration phenomenon of endemic equilibrium.

Despite, to the best of our knowledge, sufficient evidence about the important impacts of the movements of hosts and vectors under advective environments on disease transmission, it has rarely been fully studied. From a mathematical view, the main reasons may be that, on the one hand, unlike well-known SIS or SIR model, the vector-borne disease contains two infection pathways, which leads to the linearized system including two equations, so it seems difficult to derive the explicit variational characterization of $\mathfrak{R}_0$; On the other hand, population dynamics (external supplies/recruitment) are often incorporated into modeling due to the relatively short life span of vectors, and hosts and vectors possess distinct dispersal strategies [44]. These factors thereby bring certain difficulties to the analysis of $\mathfrak{R}_0$, such as asymptotic profiles, monotonicity and level set classification.

Inspired by the above discussions, the main purpose of this paper is to address the question: How does the movement of infected hosts and vectors affect the spatial dynamics of model (1.1) in advective heterogeneous environments? More precisely, the existence and stability of disease-free equilibrium (DFE) and endemic equilibrium (EE) of (1.1) are investigated by classifying the level set of $\mathfrak{R}_0$, and the aggregation phenomenon of EE is discussed, so as to explore the effects of autonomous movement and passive movement on vector-borne disease. Many interesting and important phenomena have been found in this paper, which are briefly summarized as follows:

- **(F1)** When the habitat ($\mathfrak{R}_{0a}^{\mathrm{loc}} > 1$) and the downstream end (Case (H1)) are located in a high-risk and low-risk site, respectively, there are unique surfaces dependent on diffusion rates D_I and d_I, such that the DFE is globally asymptotically stable (g.a.s.) if advection rates c_1 or c_2 is above the surface, namely, the disease will disappear; System (1.1) is uniformly persistent and admits at least one EE if c_1 or c_2 is below the surface, which implies that the disease will break out (see Theorem 4.1 and Fig. 2).
- **(F2)** When the habitat ($\mathfrak{R}_{0a}^{\mathrm{loc}} < 1$) and the downstream end (Case (H1)) are both located in low-risk sites, there are two critical points $\widetilde{D}_I$ and $\widetilde{d}_I$, such that, in the region $(0, \widetilde{D}_I) \times (0, \widetilde{d}_I)$, there exist unique surfaces, so that the DFE is g.a.s. if c_1 or c_2 is above the surface; System (1.1) is uniformly persistent and admits at least one EE if c_1 or c_2 is below the surface; In the region $[\widetilde{D}_I, \infty) \times [\widetilde{d}_I, \infty)$, the DFE is always g.a.s. for any $c_1 > 0$ and $c_2 > 0$ which indicates that the disease will be eliminated no matter how large the diffusion rates are (see Theorem 4.2 (I) and Fig. 3).
- **(F3)** When the habitat ($\mathfrak{R}_{0a}^{\mathrm{loc}} < 1$) and the downstream end (Case (H2)) are located in a low-risk and high-risk site, respectively, there are two critical points $\widetilde{D}_I$ and $\widetilde{d}_I$, such that, in the region $(0, \widetilde{D}_I] \times (0, \widetilde{d}_I]$, system (1.1) is uniformly persistent and admits at least one EE for any $c_1 > 0$ and $c_2 > 0$; In the region $(\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)$, there exist unique surfaces which are monotonically increasing w.r.t. D_I and d_I, respectively, such that system (1.1) is uniformly persistent and admits at least one EE if c_1 or c_2 is above the surface; The DFE is g.a.s. if c_1 or c_2 is below the surface (see Theorem 4.2 (II) and Fig. 4).
- **(F4)** When the downstream end (Case (H2)) is located in a high-risk site, if the advection rates of infected hosts and vectors are large relative to their diffusion rates, in other words, the

advection effect is dominant in the migration, then there exists at least one EE, and the infected hosts or vectors will aggregate downstream end (see Theorems 5.1-5.2).

Together with (F1)-(F4), we see that the spatial dynamics of vector-borne diseases become more complex and abundant in advective heterogeneous environments. Our findings not only complement the results in non-advective environments [4,23,41,44], but also may provide new perspectives for the investigation and control of the disease. Compared with the existing literatures, it should be pointed out that the main innovation of this paper lies in that it is perhaps the first time that the advective heterogeneous environments are included in vector-borne disease modeling, and the level set of basic reproduction ratio w.r.t. diffusion and advection rates is classified in detail. It is believed that these conclusions seem to lead to a deeper understanding of the underlying dynamics of vector-borne diseases.

On account of the no-flux boundary conditions and the complexity of linearized system, some crucial improvements are necessary, and major improvements are listed as follows. (1) By means of an ingenious transformation, we transform the no-flux boundary conditions into homogeneous Neumann boundary conditions, which allows us to utilize the classical comparison principle of parabolic systems to cope with the well-posedness and uniform persistence of model (1.1) (see Theorems 2.1 and 2.2). (2) Generally, the global attractivity of DFE when $\mathfrak{R}_0 < 1$ is usually solved with the aid of an auxiliary system [3,19,38,42], and the attractivity when $\mathfrak{R}_0 = 1$ is rarely addressed. Fortunately, via constructing an appropriate Lyapunov functional and employing the LaSalle's invariance principle [14], we obtain the attractivity of DFE when $\mathfrak{R}_0 \le 1$ (see Lemma 2.4). (3) With the help of the strong maximum principle of elliptic equations [29], we overcome the technical difficulties to address the aggregation effect (see Lemma 5.1), which generalizes the results involving a single elliptic equation in [5, Lemma 3.1].

The remainder of the paper is organized as follows. Section 2 studies the well-posedness and threshold dynamics of model (1.1). Section 3 explores the asymptotic profiles and monotonicity of basic reproduction ratio. The level set of basic reproduction ratio is classified in Section 4, and the aggregation phenomenon is investigated in Section 5. Section 6 gives a brief discussion to conclude the article.

2. Global dynamics

In this section, we deal with the well-posedness of (1.1), define the basic reproduction ratio $\mathfrak{R}_0$, and then study the threshold dynamics in terms of $\mathfrak{R}_0$.

2.1. Well-posedness and basic reproduction ratio

Let $\mathbb{X} := C([0, L], \mathbb{R}^5)$ be endowed with the supreme norm, and $\mathbb{X}^+ := C([0, L], \mathbb{R}^5_+)$ be the positive cone of $\mathbb{X}$. Throughout this paper, denoting $\|\cdot\| := \|\cdot\|_{L^\infty((0,L))}$ and

$$\varphi := (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5), \;\; \mathbf{u} := (S_h, I_h, R_h, S_v, I_v),$$

and

$$h^+ := \max\{h(x), x \in [0, L]\}, \;\; h^- := \min\{h(x), x \in [0, L]\},$$

here $h(\cdot)$ represents the coefficients of (1.1). The well-posedness of system (1.1) are stated in the following theorem.

Theorem 2.1. (Well-posedness) *Assume that (A1) holds. For any $\varphi \in \mathbb{X}^+$, system (1.1) has a unique nonnegative solution $\mathbf{u}(t, \cdot; \varphi)$ on $[0, \infty) \times [0, L]$ which eventually lies in the region*

$$\Omega := \left\{ \mathbf{u} \in \mathbb{X}^+ \middle| \begin{array}{l} 0 \le S_h \le C_1 e^{\frac{c_1}{D_S} L}, 0 \le I_h \le C_2 e^{\frac{c_1}{D_I} L}, 0 \le R_h \le C_3 e^{\frac{c_1}{D_R} L}, \\ 0 \le S_v \le C_4 e^{\frac{c_2}{d_S} L}, 0 \le I_v \le C_5 e^{\frac{c_2}{d_I} L} \end{array} \right\},$$

for $C_i > 0$, $i = 1, \cdots, 5$, which will be determined later. Furthermore, the solution semiflow $P(t)\varphi := \mathbf{u}(t, \cdot; \varphi)$ admits a global compact attractor in $\mathbb{X}^+$.

Proof. Taking a transformation

$$(S_h, I_h, R_h, S_v, I_v) = \left(e^{\frac{c_1}{D_S} x} \overline{S}_h, e^{\frac{c_1}{D_I} x} \overline{I}_h, e^{\frac{c_1}{D_R} x} \overline{R}_h, e^{\frac{c_2}{d_S} x} \overline{S}_v, e^{\frac{c_2}{d_I} x} \overline{I}_v \right), \tag{2.1}$$

and substituting it into system (1.1) to yield

$$\begin{cases} \overline{S}_{ht} = D_S \overline{S}_{hxx} + c_1 \overline{S}_{hx} + \Lambda_1(x) - g_{11}(x) - \mu_h(x) \overline{S}_h, & t > 0, \ x \in (0, L), \\ \overline{I}_{ht} = D_I \overline{I}_{hxx} + c_1 \overline{I}_{hx} + g_{12}(x) - \alpha_h(x) \overline{I}_h, & t > 0, \ x \in (0, L), \\ \overline{R}_{ht} = D_R \overline{R}_{hxx} + c_1 \overline{R}_{hx} + \bar{\gamma}_h(x) \overline{I}_h - \mu_h(x) \overline{R}_h, & t > 0, \ x \in (0, L), \\ \overline{S}_{vt} = d_S \overline{S}_{vxx} + c_2 \overline{S}_{vx} + M_1(x) - g_{21}(x) - \mu_v(x) \overline{S}_v, & t > 0, \ x \in (0, L), \\ \overline{I}_{vt} = d_I \overline{I}_{vxx} + c_2 \overline{I}_{vx} + g_{22}(x) - \mu_v(x) \overline{I}_v, & t > 0, \ x \in (0, L), \\ \overline{S}_{hx} = \overline{I}_{hx} = \overline{R}_{hx} = \overline{S}_{vx} = \overline{I}_{vx} = 0, & t > 0, \ x = 0, L, \\ \overline{S}_h(0, x) = e^{-\frac{c_1}{D_S} x} \varphi_1(x), \overline{I}_h(0, x) = e^{-\frac{c_1}{D_I} x} \varphi_2(x), \overline{R}_h(0, x) = e^{-\frac{c_1}{D_R} x} \varphi_3(x), & x \in (0, L), \\ \overline{S}_v(0, x) = e^{-\frac{c_2}{d_S} x} \varphi_4(x), \overline{I}_v(0, x) = e^{-\frac{c_2}{d_I} x} \varphi_5(x), & x \in (0, L), \end{cases} \tag{2.2}$$

where $\Lambda_1(x) = e^{-\frac{c_1}{D_S} x} \Lambda(x)$, $M_1(x) = e^{-\frac{c_2}{d_S} x} M(x)$, $\bar{\gamma}_h(x) = e^{\left(\frac{c_1}{D_I} - \frac{c_1}{D_R}\right) x} \gamma_h(x)$ and

$$g_{11}(x) = \frac{\beta_1(x) \overline{S}_h f_1(x, e^{\frac{c_2}{d_I} x} \overline{I}_v)}{e^{\frac{c_1}{D_S} x} \overline{S}_h + e^{\frac{c_1}{D_I} x} \overline{I}_h + e^{\frac{c_1}{D_R} x} \overline{R}_h}, \quad g_{12}(x) = \frac{\beta_1(x) e^{\left(\frac{c_1}{D_S} - \frac{c_1}{D_I}\right) x} \overline{S}_h f_1(x, e^{\frac{c_2}{d_I} x} \overline{I}_v)}{e^{\frac{c_1}{D_S} x} \overline{S}_h + e^{\frac{c_1}{D_I} x} \overline{I}_h + e^{\frac{c_1}{D_R} x} \overline{R}_h},$$

and

$$g_{21}(x) = \frac{\beta_2(x) \overline{S}_v f_2(x, e^{\frac{c_1}{D_I} x} \overline{I}_h)}{e^{\frac{c_2}{d_S} x} \overline{S}_v + e^{\frac{c_2}{d_I} x} \overline{I}_v}, \quad g_{22}(x) = \frac{\beta_2(x) e^{\left(\frac{c_2}{d_S} - \frac{c_2}{d_I}\right) x} \overline{S}_v f_2(x, e^{\frac{c_1}{D_I} x} \overline{I}_h)}{e^{\frac{c_2}{d_S} x} \overline{S}_v + e^{\frac{c_2}{d_I} x} \overline{I}_v}.$$

For any $\overline{\varphi} := (\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3, \overline{\varphi}_4, \overline{\varphi}_5) = (e^{-\frac{c_1}{D_S}x}\varphi_1, e^{-\frac{c_1}{D_I}x}\varphi_2, e^{-\frac{c_1}{D_R}x}\varphi_3, e^{-\frac{c_2}{d_S}x}\varphi_4, e^{-\frac{c_2}{d_I}x}\varphi_5) \in \mathbb{X}^+$, by using [44, Lemma 1] and the strong maximum principle, there exists a constant $T_m \le \infty$ such that system (2.2) admits a unique positive solution $(\overline{S}_h, \overline{I}_h, \overline{R}_h, \overline{S}_v, \overline{I}_v)$ on $[0, T_m) \times (0, L)$.

Consider the following system

$$\begin{cases} \widetilde{S}_{ht} = D_S \widetilde{S}_{hxx} + c_1 \widetilde{S}_{hx} + \Lambda_1(x) - \mu_h(x)\widetilde{S}_h, & t > 0,\ x \in (0, L), \\ \widetilde{S}_{hx}(t, 0) = \widetilde{S}_{hx}(t, L) = 0, & t > 0. \end{cases}$$

By applying the ideas of [42, Lemma 2.1] and [10, Theorem 2.2], we can prove the above system has a unique positive steady state $H(x)$ which is globally attractive in $C([0, L], \mathbb{R})$. Thus, the comparison principle yields that $\limsup_{t\to\infty} \overline{S}_h(t, \cdot) \le \limsup_{t\to\infty} \widetilde{S}_h(t, \cdot) = H(\cdot)$ uniformly in $(0, L)$. Then there is a constant $C_1 = \|H(\cdot)\| > 0$, independent of $\overline{\varphi}$, such that

$$\|\overline{S}_h(t, x)\| \le C_1, \quad \text{for any } \overline{\varphi} \in \mathbb{X}^+,\ t \in [0, T_m). \tag{2.3}$$

Similarly, there exist constants $C_2 = \|\overline{H}(x)\| > 0$, $C_3 = \|A(x)\| > 0$, $C_4 = \|W(x)\| > 0$ and $C_5 = \|\overline{W}(x)\| > 0$, independent of $\overline{\varphi}$, such that

$$\|\overline{I}_h(t, x)\| \le C_2,\ \|\overline{R}_h(t, x)\| \le C_3,\ \|\overline{S}_v(t, x)\| \le C_4 \text{ and } \|\overline{I}_v(t, x)\| \le C_5, \tag{2.4}$$

for any $\overline{\varphi} \in \mathbb{X}^+$, $t \in [0, T_m)$ and where $\overline{H}(\cdot)$, $A(\cdot)$, $W(\cdot)$ and $\overline{W}(\cdot)$ are the steady states of the following systems respectively:

$$\begin{cases} \widetilde{I}_{ht} = D_I \widetilde{I}_{hxx} + c_1 \widetilde{I}_{hx} + b_1\beta_1(x)e^{-\frac{c_1}{D_I}x} - \alpha_h(x)\widetilde{I}_h, & t > 0,\ x \in (0, L), \\ \widetilde{I}_{hx}(t, 0) = \widetilde{I}_{hx}(t, L) = 0, & t > 0, \end{cases}$$

$$\begin{cases} \widetilde{R}_{ht} = D_R \widetilde{R}_{hxx} + c_1 \widetilde{R}_{hx} + \bar{\gamma}_h^+ C_2 - \mu_h(x)\widetilde{R}_h, & t > 0,\ x \in (0, L), \\ \widetilde{R}_{hx}(t, 0) = \widetilde{R}_{hx}(t, L) = 0, & t > 0, \end{cases}$$

and

$$\begin{cases} \widetilde{S}_{vt} = d_S \widetilde{S}_{vxx} + c_2 \widetilde{S}_{vx} + M_1(x) - \mu_v(x)\widetilde{S}_v, & t > 0,\ x \in (0, L), \\ \widetilde{S}_{vx}(t, 0) = \widetilde{S}_{vx}(t, L) = 0, & t > 0, \end{cases}$$

$$\begin{cases} \widetilde{I}_{vt} = d_I \widetilde{I}_{vxx} + c_2 \widetilde{I}_{vx} + b_2\beta_2(x)e^{-\frac{c_2}{d_I}x} - \mu_v(x)\widetilde{I}_v, & t > 0,\ x \in (0, L), \\ \widetilde{I}_{vx}(t, 0) = \widetilde{I}_{vx}(t, L) = 0, & t > 0, \end{cases}$$

where $b_i = \max\{f_i(x, I) : x \in [0, L], I > 0\} < +\infty$, $i = 1, 2$, by the assumption (A1).

Therefore, it follows from (2.3) and (2.4) that the solution of (2.2) exists globally on $[0, \infty) \times [0, L]$, and lies in the invariant region $\overline{\Omega}$ eventually, wherein

$$\overline{\Omega} := \left\{ (\overline{S}_h, \overline{I}_h, \overline{R}_h, \overline{S}_v, \overline{I}_v) \in \mathbb{X}^+ \,\middle|\, \begin{matrix} 0 \le \overline{S}_h \le C_1, 0 \le \overline{I}_h \le C_2, 0 \le \overline{R}_h \le C_3, \\ 0 \le \overline{S}_v \le C_4, 0 \le \overline{I}_v \le C_5 \end{matrix} \right\}.$$

According to the transformation (2.1), system (1.1) has a global solution $\mathbf{u}$ on $[0,\infty)\times[0,L]$, and lies in the invariant region Ω eventually. Then $\mathbf{u}$ is ultimately bounded which indicates that system (1.1) is point dissipative and the semiflow $P(t)$ is compact. Hence, the $P(t)$ admits a compact global attractor in $\mathbb{X}^+$ by applying [15, Theorem 3.4.8]. Moreover, based on the above discussions, system (2.2) admits a unique DFE $\overline{E}_0=(H(x),0,0,W(x),0)$, and equivalently system (1.1) has a unique DFE $E_0=(H_1(x),0,0,W_1(x),0)$ satisfying $H_1(x)=e^{c_1x/D_S}H(x)$ and $W_1(x)=e^{c_2x/d_S}W(x)$, $x\in[0,L]$. □

Linearizing system (1.1) at E_0 to get

$$\begin{cases} \widehat{I}_{ht}=D_I\widehat{I}_{hxx}-c_1\widehat{I}_{hx}+\hat{\beta}_1(x)\widehat{I}_v-\alpha_h(x)\widehat{I}_h, & t>0,\ x\in(0,L),\\ \widehat{I}_{vt}=d_I\widehat{I}_{vxx}-c_2\widehat{I}_{vx}+\hat{\beta}_2(x)\widehat{I}_h-\mu_v(x)\widehat{I}_v, & t>0,\ x\in(0,L),\\ D_I\widehat{I}_{hx}-c_1\widehat{I}_h=d_I\widehat{I}_{vx}-c_2\widehat{I}_v=0, & t>0,\ x=0,L, \end{cases} \tag{2.5}$$

where $\hat{\beta}_1(\cdot)=\beta_1(\cdot)\partial_{I_v}f_1(\cdot,0)$ and $\hat{\beta}_2(\cdot)=\beta_2(\cdot)\partial_{I_h}f_2(\cdot,0)$. Define the operators $\mathbf{F},\mathbf{B}:C([0,L],\mathbb{R}^2)\to C([0,L],\mathbb{R}^2)$ by

$$\mathbf{F}(x)=\begin{pmatrix} 0 & \hat{\beta}_1(x)\\ \hat{\beta}_2(x) & 0\end{pmatrix},\quad -\mathbf{B}(x)=\begin{pmatrix} D_I\partial_x^2-c_1\partial_x-\alpha_h(x) & 0\\ 0 & d_I\partial_x^2-c_2\partial_x-\mu_v(x)\end{pmatrix},$$

wherein ∂_x and ∂_x^2 denote the first and second derivatives w.r.t. x, respectively. Let $\mathfrak{L}[\upsilon](x):=\int_0^\infty \mathbf{F}(x)\widetilde{T}(t)\upsilon(x)\mathrm{d}t$, here $\upsilon(x)$ is assumed to be the initial density distribution of infected hosts and vectors at $x\in(0,L)$, and $\widetilde{T}(t)$ is the semigroup generated by $\mathrm{d}\upsilon/\mathrm{d}t=-\mathbf{B}\upsilon$ subject to the no-flux boundary condition. Through utilizing the next generation operator approach of [36], the basic reproduction ratio of (1.1) is defined by the spectral radius of $\mathfrak{L}$, i.e.,

$$\mathfrak{R}_0(D_I,d_I,c_1,c_2):=r(\mathfrak{L}).$$

Hence, we have the following characterization of $\mathfrak{R}_0:=\mathfrak{R}_0(D_I,d_I,c_1,c_2)$.

Lemma 2.1. *Assume that (A1)-(A2) hold. Let $\kappa_0:=\kappa_0(D_I,d_I,c_1,c_2)$ be the positive eigenvalue of the elliptic eigenvalue problem*

$$\begin{cases} -D_I\psi_{2xx}+c_1\psi_{2x}+\alpha_h(x)\psi_2=\kappa\hat{\beta}_1(x)\psi_4, & x\in(0,L),\\ -d_I\psi_{4xx}+c_2\psi_{4x}+\mu_v(x)\psi_4=\kappa\hat{\beta}_2(x)\psi_2, & x\in(0,L),\\ -D_I\psi_{2x}+c_1\psi_2=-d_I\psi_{4x}+c_2\psi_4=0, & x=0,L, \end{cases} \tag{2.6}$$

with a positive eigenfunction. Then κ_0 is unique and $\mathfrak{R}_0=1/\kappa_0$.

Proof. Assume that (ψ_2,ψ_4) is the positive eigenfunction corresponding to κ_0 of problem (2.6), and let $(\psi_2,\psi_4)=e^{\sigma x}(\phi_2,\phi_4)$. By simple calculations, (ϕ_2,ϕ_4) fulfills

$$\begin{cases} -D_I\phi_{2xx}-c_1\phi_{2x}+\alpha_h(x)\phi_2=\kappa_0\hat{\beta}_1(x)\phi_4, & x\in(0,L),\\ -d_I\phi_{4xx}-c_2\phi_{4x}+\mu_v(x)\phi_4=\kappa_0\hat{\beta}_2(x)\phi_2, & x\in(0,L),\\ \phi_{2x}(0)=\phi_{2x}(L)=\phi_{4x}(0)=\phi_{4x}(L)=0. \end{cases} \tag{2.7}$$

To prove $\mathfrak{R}_0 = 1/\kappa_0$, inspired by the arguments of [36, Theorem 3.2], it is necessary to show the uniqueness of κ_0. By means of Lemma 2.2 in [27], we suppose that $\bar{\kappa}_0$ is another eigenvalue with positive eigenfunction $(\bar{\phi}_2, \bar{\phi}_4)$ such that

$$\begin{cases} -D_I\bar{\phi}_{2xx} - c_1\bar{\phi}_{2x} + \alpha_h(x)\bar{\phi}_2 = \bar{\kappa}_0\hat{\beta}_2(x)\bar{\phi}_4, & x \in (0, L), \\ -d_I\bar{\phi}_{4xx} - c_2\bar{\phi}_{4x} + \mu_v(x)\bar{\phi}_4 = \bar{\kappa}_0\hat{\beta}_1(x)\bar{\phi}_2, & x \in (0, L), \\ \bar{\phi}_{2x}(0) = \bar{\phi}_{2x}(L) = \bar{\phi}_{4x}(0) = \bar{\phi}_{4x}(L) = 0. \end{cases} \tag{2.8}$$

Multiplying the first equation of (2.6) and (2.8) by $\bar{\phi}_2$ and ψ_2 respectively, and then integrating by parts over $(0, L)$, one obtains

$$\begin{cases} D_I\int_0^L \bar{\phi}_{2x}\psi_{2x}\mathrm{d}x - c_1\int_0^L \psi_2\bar{\phi}_{2x}\mathrm{d}x + \int_0^L \alpha_h(x)\psi_2\bar{\phi}_2\mathrm{d}x = \kappa_0\int_0^L \hat{\beta}_1(x)\psi_4\bar{\phi}_2\mathrm{d}x, \\ D_I\int_0^L \bar{\phi}_{2x}\psi_{2x}\mathrm{d}x - c_1\int_0^L \psi_2\bar{\phi}_{2x}\mathrm{d}x + \int_0^L \alpha_h(x)\psi_2\bar{\phi}_2\mathrm{d}x = \bar{\kappa}_0\int_0^L \hat{\beta}_2(x)\psi_2\bar{\phi}_4\mathrm{d}x. \end{cases}$$

Subtracting the above two equations to obtain

$$\kappa_0\int_0^L \hat{\beta}_1(x)\psi_4\bar{\phi}_2\mathrm{d}x - \bar{\kappa}_0\int_0^L \hat{\beta}_2(x)\psi_2\bar{\phi}_4\mathrm{d}x = 0. \tag{2.9}$$

Similarly, we multiply the second equation of (2.6) and (2.8) by $\bar{\phi}_4$ and ψ_4 respectively, and integrate by parts in $(0, L)$, and then subtract the resulting equations to get

$$\kappa_0\int_0^L \hat{\beta}_2(x)\psi_2\bar{\phi}_4\mathrm{d}x - \bar{\kappa}_0\int_0^L \hat{\beta}_1(x)\psi_4\bar{\phi}_2\mathrm{d}x = 0. \tag{2.10}$$

Then, adding (2.9) and (2.10), one has

$$(\kappa_0 - \bar{\kappa}_0)\left[\int_0^L \hat{\beta}_1(x)\psi_4\bar{\phi}_2\mathrm{d}x + \int_0^L \hat{\beta}_2(x)\psi_2\bar{\phi}_4\mathrm{d}x\right] = 0.$$

By the positivity of $\hat{\beta}_i$, ψ_j and $\bar{\phi}_j$ on $[0, L]$, $i = 1, 2$, $j = 2, 4$, thus $\kappa_0 = \bar{\kappa}_0$ which means that κ_0 is unique. Then $\mathfrak{R}_0 = 1/\kappa_0$ by ideas similar to those in [36, Theorem 3.2 and Remark 3.1] and the uniqueness of κ_0 with the positive eigenfunction. □

Lemma 2.2. *Assume that (A1) holds. For $D_I > 0$, $d_I > 0$, $c_1 > 0$ and $c_2 > 0$, then the following inequality is valid:*

$$\sqrt{\frac{\hat{\beta}_1^- \hat{\beta}_2^-}{\alpha_h^+ \mu_v^+}} \leq \Re_0(D_I, d_I, c_1, c_2) \leq \sqrt{\frac{\hat{\beta}_1^+ \hat{\beta}_2^+}{\alpha_h^- \mu_v^-}}.$$

Proof. By Lemma 2.1, $1/\Re_0$ is the principle eigenvalue of (2.6), i.e.,

$$\begin{cases} -D_I \psi_{2xx} + c_1 \psi_{2x} + \alpha_h(x)\psi_2 = \dfrac{1}{\Re_0}\hat{\beta}_1(x)\psi_4, & x \in (0, L), \\ -d_I \psi_{4xx} + c_2 \psi_{4x} + \mu_v(x)\psi_4 = \dfrac{1}{\Re_0}\hat{\beta}_2(x)\psi_2, & x \in (0, L), \\ -D_I \psi_{2x} + c_1 \psi_2 = -d_I \psi_{4x} + c_2 \psi_4 = 0, & x = 0, L. \end{cases}$$

Integrating the two equations of above system by parts over $(0, L)$, one has

$$\begin{cases} \displaystyle\int_0^L \alpha_h(x)\psi_2 \mathrm{d}x = \frac{1}{\Re_0} \int_0^L \hat{\beta}_1(x)\psi_4 \mathrm{d}x, \\ \displaystyle\int_0^L \mu_v(x)\psi_4 \mathrm{d}x = \frac{1}{\Re_0} \int_0^L \hat{\beta}_2(x)\psi_2 \mathrm{d}x. \end{cases}$$

Thus,

$$\sqrt{\frac{\hat{\beta}_1^- \hat{\beta}_2^-}{\alpha_h^+ \mu_v^+}} \leq \Re_0 = \sqrt{\frac{\int_0^L \hat{\beta}_1(x)\psi_4 \mathrm{d}x \int_0^L \hat{\beta}_2(x)\psi_2 \mathrm{d}x}{\int_0^L \alpha_h(x)\psi_2 \mathrm{d}x \int_0^L \mu_v(x)\psi_4 \mathrm{d}x}} \leq \sqrt{\frac{\hat{\beta}_1^+ \hat{\beta}_2^+}{\alpha_h^- \mu_v^-}}$$

which establishes the boundedness of $\Re_0$. □

Denoting $(\widehat{I_h}(t, \cdot), \widehat{I_v}(t, \cdot)) = e^{-\kappa t}(\eta_2(\cdot), \eta_4(\cdot))$ and substituting it into the linearized system (2.5) to yield

$$\begin{cases} D_I \eta_{2xx} - c_1 \eta_{2x} + \hat{\beta}_1(x)\eta_4 - \alpha_h(x)\eta_2 + \kappa \eta_2 = 0, & x \in (0, L), \\ d_I \eta_{4xx} - c_2 \eta_{4x} + \hat{\beta}_2(x)\eta_2 - \mu_v(x)\eta_4 + \kappa \eta_4 = 0, & x \in (0, L), \\ D_I \eta_{2x} - c_1 \eta_2 = d_I \eta_{4x} - c_2 \eta_4 = 0, & x = 0, L. \end{cases}$$

Let $(\eta_2, \eta_4) = e^{\sigma x}(p_2, p_4)$. Then (p_2, p_4) satisfies

$$\begin{cases} D_I p_{2xx} + c_1 p_{2x} + \hat{\beta}_1(x)p_4 - \alpha_h(x)p_2 + \kappa p_2 = 0, & x \in (0, L), \\ d_I p_{4xx} + c_2 p_{4x} + \hat{\beta}_2(x)p_2 - \mu_v(x)p_4 + \kappa p_4 = 0, & x \in (0, L), \\ p_{2x}(0) = p_{2x}(L) = p_{4x}(0) = p_{4x}(L) = 0. \end{cases} \tag{2.11}$$

It thus from the Krein-Rutman theorem [18] that problem (2.11) has a unique principal eigenvalue $\kappa_1 := \kappa_1(D_I, d_I, c_1, c_2)$, i.e., κ_1 is real and simple with positive eigenfunction (p_2, p_4) and the real parts of other eigenvalues are strictly greater than κ_1.

Lemma 2.3. *Assume that (A1)-(A2) hold. For $D_I > 0$, $d_I > 0$, $c_1 > 0$ and $c_2 > 0$, then $\kappa_1 < 0$ if $\mathfrak{R}_0 > 1$, $\kappa_1 = 0$ if $\mathfrak{R}_0 = 1$, and $\kappa_1 > 0$ if $\mathfrak{R}_0 < 1$.*

Proof. Set $(\bar{p}_2, \bar{p}_4)$ be the positive eigenfunction of the corresponding eigenvalue κ_1 for the adjoint problem of (2.11). Then $(\bar{p}_2, \bar{p}_4)$ meets

$$\begin{cases} -D_I\bar{p}_{2xx} - c_1\bar{p}_{2x} - \hat{\beta}_2(x)\bar{p}_4 + \alpha_h(x)\bar{p}_2 = \kappa_1\bar{p}_2, & x \in (0, L), \\ -d_I\bar{p}_{4xx} - c_2\bar{p}_{4x} - \hat{\beta}_1(x)\bar{p}_2 + \mu_v(x)\bar{p}_4 = \kappa_1\bar{p}_4, & x \in (0, L), \\ \bar{p}_{2x}(0) = \bar{p}_{2x}(L) = \bar{p}_{4x}(0) = \bar{p}_{4x}(L) = 0. \end{cases} \tag{2.12}$$

Multiplying the first equation of (2.6) and (2.12) by $\bar{p}_2$ and ψ_2 respectively, and then integrating by parts over $(0, L)$, one gets

$$\begin{cases} D_I\int_0^L \bar{p}_{2x}\psi_{2x}\mathrm{d}x - c_1\int_0^L \psi_2\bar{p}_{2x}\mathrm{d}x + \int_0^L \alpha_h\bar{p}_2\psi_2\mathrm{d}x = \frac{1}{\mathfrak{R}_0}\int_0^L \hat{\beta}_1\bar{p}_2\psi_4\mathrm{d}x, \\ D_I\int_0^L \bar{p}_{2x}\psi_{2x}\mathrm{d}x - c_1\int_0^L \psi_2\bar{p}_{2x}\mathrm{d}x - \int_0^L \hat{\beta}_2\bar{p}_4\psi_2\mathrm{d}x + \int_0^L \alpha_h\bar{p}_2\psi_2\mathrm{d}x = \kappa_1\int_0^L \bar{p}_2\psi_2\mathrm{d}x. \end{cases}$$

Subtracting the above two equations to yield

$$\kappa_1\int_0^L \bar{p}_2\psi_2\mathrm{d}x = \frac{1}{\mathfrak{R}_0}\int_0^L \hat{\beta}_1(x)\bar{p}_2\psi_4\mathrm{d}x - \int_0^L \hat{\beta}_2(x)\bar{p}_4\psi_2\mathrm{d}x. \tag{2.13}$$

Similarly, multiplying the second equation of (2.6) and (2.12) by $\bar{p}_4$ and ψ_4 respectively, integrating by parts over $(0, L)$, and then subtracting two resulting equalities, one has

$$\kappa_1\int_0^L \bar{p}_4\psi_4\mathrm{d}x = \frac{1}{\mathfrak{R}_0}\int_0^L \hat{\beta}_2(x)\bar{p}_4\psi_2\mathrm{d}x - \int_0^L \hat{\beta}_1(x)\bar{p}_2\psi_4\mathrm{d}x. \tag{2.14}$$

Adding (2.13) and (2.14) to get

$$\kappa_1\left(\int_0^L \bar{p}_2\psi_2\mathrm{d}x + \int_0^L \bar{p}_4\psi_4\mathrm{d}x\right) = \left(\frac{1}{\mathfrak{R}_0} - 1\right)\left(\int_0^L \hat{\beta}_1(x)\bar{p}_2\psi_4\mathrm{d}x + \int_0^L \hat{\beta}_2(x)\bar{p}_4\psi_2\mathrm{d}x\right).$$

Since ψ_j, $\bar{p}_j$ and $\hat{\beta}_i$ are positive on $(0, L)$, $i = 1, 2$, $j = 2, 4$, we have $sign(1 - \mathfrak{R}_0) = sign\kappa_1$. □

2.2. Threshold dynamics

In this subsection, the global stability of E_0 when $\Re_0 < 1$, uniform persistence and existence of EE for (1.1) when $\Re_0 > 1$ are investigated. We have the following results:

Theorem 2.2. (**Stability and persistence**) *Assume that (A1)-(A2) hold. If* $\Re_0 < 1$*, then* E_0 *is g.a.s. If* $\Re_0 > 1$*, then there exists a constant* $\delta_0 > 0$*, such that the solution of* (1.1) *meets*

$$\liminf_{t\to\infty} \|(S_h(t,x), I_h(t,x), R_h(t,x), S_v(t,x), I_v(t,x)) - (H_1(x), 0, 0, W_1(x), 0)\| > \delta_0 \quad (2.15)$$

uniformly for $x \in [0, L]$*. Moreover, system* (1.1) *admits at least one EE.*

We will complete the proof of Theorem 2.2 via proving several lemmas.

Lemma 2.4. *Assume that (A1)-(A2) hold and* $\Re_0 \le 1$*. Then* E_0 *is globally attractive, that is,*

$$\lim_{t\to\infty} (S_h(t,x), I_h(t,x), R_h(t,x), S_v(t,x), I_v(t,x)) = (H_1(x), 0, 0, W_1(x), 0)$$

uniformly for $x \in [0, L]$.

Proof. Let $P(t)\varphi = (S_h(t,\cdot), I_h(t,\cdot), R_h(t,\cdot), S_v(t,\cdot), I_v(t,\cdot))$ be the unique solution of (1.1) with $\varphi \in \Omega$, where the set Ω is determined by Theorem 2.1. To prove $(I_h(t,\cdot), I_v(t,\cdot)) \to (0,0)$ as $t \to \infty$ in $[0, L]$. According to [27], thanks to the Sobolev inequalities and L^p estimates, for any $\beta \in (0,1)$, there is a constant $C_5 > 0$ such that

$$\|(S_h, I_h, R_h, S_v, I_v)\|_{C^{\frac{\beta}{2},\beta}([t^*-1,t^*+1]\times[0,L])} \le C_5 \|(S_h, I_h, R_h, S_v, I_v)\|_{L^\infty([t^*-1,t^*+1]\times[0,L])},$$

for each $t^* \ge 1$. Then there is a constant $C_6 > 0$ such that

$$\|(S_h, I_h, R_h, S_v, I_v)\|_{C^\beta([0,L])} \le C_5 C_6, \quad \text{for any } t \ge 1.$$

Hence, $P(t)$ is compact, and for each $\varphi \in \Omega$, the orbit of $P(t)\varphi$ under the dynamical system generated by (1.1) has a compact closure in Ω.

Define a Lyapunov functional as follows

$$\mathcal{G}[\mathbf{u}](t) = \int_0^L (I_h \bar{p}_2 + I_v \bar{p}_4) \mathrm{d}x, \quad \mathbf{u} = (S_h, I_h, R_h, S_v, I_v) \in \Omega,$$

wherein $(\bar{p}_2, \bar{p}_4)$ is the positive eigenfunction corresponding to the eigenvalue κ_1 of (2.12). After elementary computations, by the second and fifth equations of (1.1) and the assumption (A1), we obtain

$$
\begin{aligned}
\dot{\mathcal{G}}[\mathbf{u}](t) &= \int_0^L (\bar{p}_2 I_{ht} + \bar{p}_4 I_{vt}) \mathrm{d}x \\
&= \int_0^L [D_I I_{hxx} - c_1 I_{hx} + g_1(x, S_h, I_h, R_h, I_v) - \alpha_h(x) I_h] \bar{p}_2 \mathrm{d}x \\
&\quad + \int_0^L [d_I I_{vxx} - c_2 I_{vx} + g_2(x, S_v, I_v, I_h) - \mu_v(x) I_v] \bar{p}_4 \mathrm{d}x \\
&= \int_0^L [D_I \bar{p}_{2xx} + c_1 \bar{p}_{2x} - \alpha_h(x) \bar{p}_2] I_h \mathrm{d}x + \int_0^L g_1(x, S_h, I_h, R_h, I_v) \bar{p}_2 \mathrm{d}x \\
&\quad + \int_0^L [d_I \bar{p}_{4xx} + c_2 \bar{p}_{4x} - \mu_v(x) \bar{p}_4] I_v \mathrm{d}x + \int_0^L g_2(x, S_v, I_v, I_h) \bar{p}_4 \mathrm{d}x \\
&= -\kappa_1 \int_0^L (I_h \bar{p}_2 + I_v \bar{p}_4) \mathrm{d}x - \int_0^L [\hat{\beta}_2(x) I_h - g_2(x, S_v, I_v, I_h)] \bar{p}_4 \mathrm{d}x \\
&\quad - \int_0^L [\hat{\beta}_1(x) I_v - g_1(x, S_h, I_h, R_h, I_v)] \bar{p}_2 \mathrm{d}x \\
&\leq -\kappa_1 \int_0^L (I_h \bar{p}_2 + I_v \bar{p}_4) \mathrm{d}x - \int_0^L \left[\partial_{I_h} f_2(x,0) I_h - f_2(x, I_h)\right] \beta_2(x) \bar{p}_4 \mathrm{d}x \\
&\quad - \int_0^L \left[\partial_{I_v} f_1(x,0) I_v - f_1(x, I_v)\right] \beta_1(x) \bar{p}_2 \mathrm{d}x \\
&\leq -\kappa_1 \int_0^L (I_h \bar{p}_2 + I_v \bar{p}_4) \mathrm{d}x,
\end{aligned}
$$

where $\dot{}$ denotes the derivative of t. Note that $\kappa_1 \geq 0$ owing to Lemma 2.3 and $\mathfrak{R}_0 \leq 1$. Since $(\bar{p}_2, \bar{p}_4)$ and (I_h, I_v) are positive, $\dot{\mathcal{G}}(t) \leq 0$, for any $t \in [0, \infty)$. Let $\mathcal{S} := \{\mathbf{u} \in \Omega \,|\, \dot{\mathcal{G}}[\mathbf{u}](t) = 0\}$. Then the maximal invariant set of $\mathcal{S}$ is $\mathcal{S}_M = \{\mathbf{u} \in \Omega \,|\, I_h \equiv 0, I_v \equiv 0\}$. Therefore, utilizing the LaSalle's invariance principle for infinite dimensional dynamical systems [14, Theorem 1] to get $(I_h(t, \cdot), I_v(t, \cdot)) \to (0, 0)$ as $t \to \infty$ uniformly in $[0, L]$ which yields that $R_h(t, \cdot) \to 0$ as $t \to \infty$ according to the third equation of (1.1).

Moreover, similar to the arguments of [44, Proposition 1], by using the theory of internally chain transitive sets established in [46], we can show that

$$\lim_{t\to\infty}\|(S_h(t,x),S_v(t,x))-(H_1(x),W_1(x))\|=0,\ \text{ uniformly for } x\in[0,L].$$

Hence, E_0 is globally attractive. □

Lemma 2.5. *Assume that (A1)-(A2) hold. If* $\mathfrak{R}_0<1$*, then* $\overline{E}_0$ *is asymptotically stable. If* $\mathfrak{R}_0>1$*, then there exists a constant* $\overline{\delta}_0>0$*, such that the solution of* (2.2) *satisfies*

$$\liminf_{t\to\infty}\|(\overline{S}_h(t,x),\overline{I}_h(t,x),\overline{R}_h(t,x),\overline{S}_v(t,x),\overline{I}_v(t,x))-(H(x),0,0,W(x),0)\|>\overline{\delta}_0 \quad (2.16)$$

uniformly for $x\in[0,L]$*. Moreover, system* (2.2) *admits at least one EE.*

Proof. When $\mathfrak{R}_0<1$, $\overline{E}_0$ is asymptotically stable resembling the ideas of Theorem 3.1 in [36]. It then remains to deal with the persistence of (2.2) in the case of $\mathfrak{R}_0>1$. Let

$$\overline{\Omega}_0:=\{\overline{\varphi}\in\overline{\Omega}\,|\,\overline{\varphi}_2\neq 0 \text{ and } \overline{\varphi}_5\neq 0\} \text{ and } \partial\overline{\Omega}_0:=\{\overline{\varphi}\in\overline{\Omega}\,|\,\overline{\varphi}_2=0 \text{ or } \overline{\varphi}_5=0\},$$

where $\overline{\varphi}$ and $\overline{\Omega}$ are defined by Theorem 2.1. It is easy to know that $\overline{\Omega}=\overline{\Omega}_0\cup\partial\overline{\Omega}_0$, and $\overline{\Omega}_0$ and $\partial\overline{\Omega}_0$ are relatively open and closed subsets of $\overline{\Omega}$, respectively. Moreover, $\overline{\Omega}_0$ is a convex set. Set $\overline{P}(t)\overline{\varphi}$ be the unique solution of (2.2) with $\overline{\varphi}\in\overline{\Omega}$. By Theorem 2.1, $\overline{P}(t)$ admits a global compact attractor and $\overline{P}(t)\overline{\Omega}_0\subset\overline{\Omega}_0$. In addition, denote U_∂ as the maximum positively invariant set of $\overline{P}(t)$ in $\partial\overline{\Omega}_0$, i.e., $U_\partial:=\{\overline{\varphi}\in\overline{\Omega}\,|\,\overline{P}(t)\overline{\varphi}\in\partial\overline{\Omega}_0,\ t\geq 0\}$. Then $U_\partial=\{\overline{\varphi}\in\overline{\Omega}\,|\,\overline{\varphi}_2=\overline{\varphi}_5=0\}$. Let $\omega(\overline{\varphi})$ be the omega limit set of $\overline{\varphi}$ in $\overline{\Omega}$, and $\overline{U}_\partial:=\bigcup_{\{\overline{\varphi}\in U_\partial\}}\omega(\overline{\varphi})$. To end the proof, we divided it into two steps:

Step 1. Claim that $\overline{U}_\partial=\{\overline{E}_0\}$. In fact, for any $\overline{\varphi}\in U_\partial$, from the definition of U_∂, we have $\overline{I}_h(t,x)\equiv\overline{I}_v(t,x)\equiv 0$, for all $x\in[0,L]$, $t\geq 0$. Substituting it into (2.2), one gets

$$\begin{cases} \overline{S}_{ht}=D_S\overline{S}_{hxx}+c_1\overline{S}_{hx}+\Lambda_1(x)-\mu_h(x)\overline{S}_h, & t>0,\ x\in(0,L),\\ \overline{R}_{ht}=D_R\overline{R}_{hxx}+c_1\overline{R}_{hx}-\mu_h(x)\overline{R}_h, & t>0,\ x\in(0,L),\\ \overline{S}_{vt}=d_S\overline{S}_{vxx}+c_2\overline{S}_{vx}+M_1(x)-\mu_v(x)\overline{S}_v, & t>0,\ x\in(0,L),\\ \overline{S}_{hx}=\overline{R}_{hx}=\overline{S}_{vx}=0, & t>0,\ x=0,L. \end{cases}$$

Then $\overline{S}_h(t,\cdot)\to H(\cdot)$, $\overline{R}_h(t,\cdot)\to 0$ and $\overline{S}_v(t,\cdot)\to W(\cdot)$ as $t\to\infty$ uniformly in $(0,L)$. Hence, $\overline{U}_\partial=\{\overline{E}_0\}$, and $\{\overline{E}_0\}$ is an isolated and compact invariant set of $\overline{P}(t)$ restricted in U_∂.

Step 2. Claim that there exists a constant $\delta_1>0$, independent of $\overline{\varphi}$, such that

$$\limsup_{t\to\infty}\|\overline{P}(t)\overline{\varphi}-(H(\cdot),0,0,W(\cdot),0)\|>\delta_1. \quad (2.17)$$

By a contradictive argument, for any $\hat{\delta}_1>0$, there exists a $\overline{\varphi}^*=(\overline{\varphi}_1^*,\overline{\varphi}_2^*,\overline{\varphi}_3^*,\overline{\varphi}_4^*,\overline{\varphi}_5^*)$ such that

$$\limsup_{t\to\infty}\|\overline{P}(t)\overline{\varphi}^*-(H(\cdot),0,0,W(\cdot),0)\|\leq\hat{\delta}_1, \quad (2.18)$$

where $\overline{P}(t)\overline{\varphi}^*=(\overline{S}_h^*(t,\cdot),\overline{I}_h^*(t,\cdot),\overline{R}_h^*(t,\cdot),\overline{S}_v^*(t,\cdot),\overline{I}_v^*(t,\cdot))$, $t>0$.

Take $\delta_2>0$ small enough. Suppose that $\kappa_1^{\delta_2}=\kappa_1(D_I,d_I,c_1,c_2,\delta_2)$ is the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} D_I p_{2xx} + c_1 p_{2x} + \dfrac{\beta_1(x) e^{\frac{c_1}{D_S}x}(H-\delta_2)\partial_{I_v} f_1(x, e^{\frac{c_2}{d_I}x}\delta_2)}{e^{\frac{c_1}{D_S}x}(H-\delta_2) + e^{\frac{c_1}{D_I}x}\delta_2 + e^{\frac{c_1}{D_R}x}\delta_2} p_4 - \alpha_h(x) p_2 + \kappa_1^{\delta_2} p_2 = 0, \\ \quad x \in (0, L), \\ d_I p_{4xx} + c_2 p_{4x} + \dfrac{\beta_2(x) e^{\frac{c_2}{d_S}x}(W-\delta_2)\partial_{I_h} f_2(x, e^{\frac{c_1}{D_I}x}\delta_2)}{e^{\frac{c_2}{d_S}x}(W-\delta_2) + e^{\frac{c_2}{d_I}x}\delta_2} p_2 - \mu_v(x) p_4 + \kappa_1^{\delta_2} p_4 = 0, \\ \quad x \in (0, L), \\ p_{2x}(0) = p_{2x}(L) = p_{4x}(0) = p_{4x}(L) = 0, \end{cases} \tag{2.19}$$

wherein (p_2, p_4) is the corresponding positive eigenfunction in $(0, L)$. Since $\mathfrak{R}_0 > 1$, $\kappa_1 < 0$ by Lemma 2.3, here κ_1 is the eigenvalue of (2.11). Notice that $\kappa_1^{\delta_2} \to \kappa_1 < 0$ as $\delta_2 \to 0$. Thus, one can choose a sufficiently small δ_2 such that $\kappa_1^{\delta_2} < 0$. From the arbitrariness of $\hat{\delta}_1$, letting $\hat{\delta}_1 = \delta_2$. Following from (2.18) that there is a $T_0^* > 0$ such that $\overline{S}_h^*(t, \cdot) \geq H(\cdot) - \delta_2$, $\overline{S}_v^*(t, \cdot) \geq W(\cdot) - \delta_2$, $\overline{I}_h^*(t, \cdot) \leq \delta_2$, $\overline{R}_h^*(t, \cdot) \leq \delta_2$ and $\overline{I}_v^*(t, \cdot) \leq \delta_2$, for any $t \geq T_0^*$ in $[0, L]$. By inspection of the assumptions (A1)-(A2), we obtain

$$\frac{\beta_1(x) e^{\left(\frac{c_1}{D_S} - \frac{c_1}{D_I}\right)x} \overline{S}_h^* f_1(x, e^{\frac{c_2}{d_I}x}\overline{I}_v^*)}{e^{\frac{c_1}{D_S}x}\overline{S}_h^* + e^{\frac{c_1}{D_I}x}\overline{I}_h^* + e^{\frac{c_1}{D_R}x}\overline{R}_h^*} \geq \frac{\beta_1(x) e^{\frac{c_1}{D_S}x}(H-\delta_2)\partial_{I_v} f_1(x, e^{\frac{c_2}{d_I}x}\delta_2)}{e^{\frac{c_1}{D_S}x}(H-\delta_2) + e^{\frac{c_1}{D_I}x}\delta_2 + e^{\frac{c_1}{D_R}x}\delta_2} \overline{I}_v^*, \tag{2.20}$$

and

$$\frac{\beta_2(x) e^{\left(\frac{c_2}{d_S} - \frac{c_2}{d_I}\right)x} \overline{S}_v^* f_2(x, e^{\frac{c_1}{D_I}x}\overline{I}_h^*)}{e^{\frac{c_2}{d_S}x}\overline{S}_v^* + e^{\frac{c_2}{d_I}x}\overline{I}_v^*} \geq \frac{\beta_2(x) e^{\frac{c_2}{d_S}x}(W-\delta_2)\partial_{I_h} f_2(x, e^{\frac{c_1}{D_I}x}\delta_2)}{e^{\frac{c_2}{d_S}x}(W-\delta_2) + e^{\frac{c_2}{d_I}x}\delta_2} \overline{I}_h^*, \tag{2.21}$$

for all $t \geq T_0^*$ and $x \in [0, L]$.

In addition, it follows from Theorem 2.1 and the strong maximum principle that $(\overline{S}_h^*, \overline{I}_h^*, \overline{R}_h^*, \overline{S}_v^*, \overline{I}_v^*) \in Int(\mathbb{X}^+)$ (interior of $\mathbb{X}^+$). Then, there is a constant $\tau_0 > 0$ small enough, such that $\overline{I}_h^*(T_0^*, x) \geq \tau_0 p_2$ and $\overline{I}_v^*(T_0^*, x) \geq \tau_0 p_4$, $x \in (0, L)$. With the help of (2.20)-(2.21), we can testify that $(\overline{I}_h^*, \overline{I}_v^*)$ is a super-solution of the problem:

$$\begin{cases} \widetilde{I}_{ht} = D_I \widetilde{I}_{hxx} + c_1 \widetilde{I}_{hx} + \dfrac{\beta_1(x) e^{\frac{c_1}{D_S}x}(H-\delta_2)\partial_{I_v} f_1(x, e^{\frac{c_2}{d_I}x}\delta_2)}{e^{\frac{c_1}{D_S}x}(H-\delta_2) + e^{\frac{c_1}{D_I}x}\delta_2 + e^{\frac{c_1}{D_R}x}\delta_2} \widetilde{I}_v - \alpha_h(x)\widetilde{I}_h, & t > 0,\ x \in (0, L), \\ \widetilde{I}_{vt} = d_I \widetilde{I}_{vxx} + c_2 \widetilde{I}_{vx} + \dfrac{\beta_2(x) e^{\frac{c_2}{d_S}x}(W-\delta_2)\partial_{I_h} f_2(x, e^{\frac{c_1}{D_I}x}\delta_2)}{e^{\frac{c_2}{d_S}x}(W-\delta_2) + e^{\frac{c_2}{d_I}x}\delta_2} \widetilde{I}_h - \mu_v(x)\widetilde{I}_v, & t > 0,\ x \in (0, L), \\ \widetilde{I}_{hx}(t, 0) = \widetilde{I}_{hx}(t, L) = \widetilde{I}_{vx}(t, 0) = \widetilde{I}_{vx}(t, L) = 0, & t > 0, \\ \widetilde{I}_h(T_0^*, x) = \tau_0 p_2(x), \ \ \widetilde{I}_v^*(T_0^*, x) = \tau_0 p_4(x), & x \in (0, L). \end{cases} \tag{2.22}$$

From (2.19), it is straightforward to observe that $(\tau_0 e^{-\kappa_1^{\delta_2}(t-T_0^*)} p_2, \tau_0 e^{-\kappa_1^{\delta_2}(t-T_0^*)} p_4)$ is a solution of (2.22). Then, thanks to the comparison principle and the fact $\kappa_1^{\delta_2} < 0$, we have

$$\overline{I}_h^*(t,x) \geq \tau_0 e^{-\kappa_1^{\delta_2}(t-T_0^*)} p_2(x) \to \infty, \quad \overline{I}_v^*(t,x) \geq \rho_0 e^{-\kappa_1^{\delta_2}(t-T_0^*)} p_4(x) \to \infty \text{ as } t \to \infty$$

which contradicts (2.18), and so (2.17) is valid which implies that $\{\overline{E}_0\}$ is an isolated invariant set of $\overline{P}(t)$ restricted in $\overline{\Omega}$, and $W^S(\{\overline{E}_0\}) \cap \overline{\Omega}_0 = \emptyset$, where $W^S(\{\overline{E}_0\})$ represents the stable set of $\{\overline{E}_0\}$ w.r.t. $\overline{P}(t)$.

Combining **Steps 1-2** and [46, Theorem 1.3.1], $\overline{P}(t)$ is uniformly persistent for $(\overline{\Omega}, \partial\overline{\Omega}_0)$. Consequently, (2.16) holds. Furthermore, system (2.2) possesses at least one endemic steady state when $\mathfrak{R}_0 > 1$ by using the Theorem 1.3.7 in [46]. □

Proof of Theorem 2.2. According to Lemma 2.5, we see that E_0 is asymptotically stable and then is g.a.s. together with Lemma 2.4 as $\mathfrak{R}_0 < 1$. In addition, Lemma 2.5 indicates that there exists a constant $\delta_0 > 0$, such that (2.15) holds, and system (1.1) has at least one EE. Theorem 2.2 is proved. □

3. Asymptotic profiles and monotonicity of basic reproduction ratio

In this section, we discuss some important properties of basic reproduction ratio $\mathfrak{R}_0$, which are concerned with the asymptotic behaviors and monotonicity.

3.1. *For the case $c_1 = c_2 = 0$*

We consider system (1.1) with non-advective effects in this subsection. When $c_1 = c_2 = 0$, the basic reproduction ratio corresponding to (1.1) is denoted as $\widetilde{\mathfrak{R}}_0 := \widetilde{\mathfrak{R}}_0(D_I, d_I)$ which was investigated in [44]. Then there are the following statements:

Proposition 3.1. (Asymptotic profiles of $\widetilde{\mathfrak{R}}_0$, [44]) *Assume that (A1) holds. Then*

(1) $\widetilde{\mathfrak{R}}_0(D_I, d_I) \to \max\left\{\mathfrak{R}_0^{\mathrm{loc}}(x),\ x \in [0, L]\right\}$ *as $D_I \to 0$ and $d_I \to 0$;*
(2) $\widetilde{\mathfrak{R}}_0(D_I, d_I) \to \mathfrak{R}_{0a}^{\mathrm{loc}}$ *as $D_I \to \infty$ and $d_I \to \infty$.*

Proposition 3.2. (Monotonicity of $\widetilde{\mathfrak{R}}_0$, [44]) *Assume that (A1) holds. For any $D_I > 0$ and $d_I > 0$, if $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ for any $x \in (0, L)$, then $\widetilde{\mathfrak{R}}_0(D_I, d_I)$ is a monotone nonincreasing function of D_I and d_I, respectively.*

According to Propositions 3.1 and 3.2, we have:

Lemma 3.1. *Assume that (A1) holds and $\mathfrak{R}_0^{vh}(x)\mathfrak{R}_0^{hv}(x) - 1$ changes sign in $(0, L)$ and $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ for any $x \in (0, L)$. Then*

(1) *If $\mathfrak{R}_{0a}^{\mathrm{loc}} > 1$, then $\widetilde{\mathfrak{R}}_0(D_I, d_I) > 1$ for any $D_I > 0$ and $d_I > 0$;*
(2) *If $\mathfrak{R}_{0a}^{\mathrm{loc}} < 1$, then there exists a unique positive point $(\widetilde{D}_I, \widetilde{d}_I)$ such that $\widetilde{\mathfrak{R}}_0(D_I, d_I) > 1$ for $0 < D_I < \widetilde{D}_I$, $0 < d_I < \widetilde{d}_I$, and $\widetilde{\mathfrak{R}}_0(D_I, d_I) < 1$ for $D_I > \widetilde{D}_I$, $d_I > \widetilde{d}_I$.*

Proof. (1) By Proposition 3.1 (2), we have

$$\lim_{D_I \to \infty,\, d_I \to \infty} \widetilde{\mathfrak{R}}_0(D_I, d_I) = \mathfrak{R}_{0a}^{\mathrm{loc}} > 1.$$

Thus, $\widetilde{\Re}_0 > 1$ for any $D_I > 0$ and $d_I > 0$ due to the monotonicity of $\widetilde{\Re}_0$ in Proposition 3.2.

(2) Similar to the proof of (1), since $\Re_0^{vh}(x)\Re_0^{hv}(x) - 1$ changes sign, we get

$$\lim_{D_I\to\infty,\, d_I\to\infty} \widetilde{\Re}_0(D_I, d_I) = \Re_{0a}^{\mathrm{loc}} < 1,$$

$$\lim_{D_I\to 0,\, d_I\to 0} \widetilde{\Re}_0(D_I, d_I) = \max\left\{\Re_0^{\mathrm{loc}}(x),\ x\in[0,L]\right\} > 1,$$

which guarantees the existence and uniqueness of $(\widetilde{D}_I, \widetilde{d}_I)$ from Proposition 3.2. □

Remark 3.1. Lemma 3.1 indicates that when there is no advection effects, the disease will break out if the habitat is a high-risk area ($\Re_{0a}^{\mathrm{loc}} > 1$); If the habitat is a low-risk area ($\Re_{0a}^{\mathrm{loc}} < 1$), then the outbreak of disease is related to the diffusion rates of infected hosts and vectors, that is, when $0 < D_I < \widetilde{D}_I$, $0 < d_I < \widetilde{d}_I$, the disease will break out no matter how small the diffusion rates are, and when $D_I > \widetilde{D}_I$, $d_I > \widetilde{d}_I$, the disease will be eliminated no matter how large the diffusion rates are.

3.2. For the case $c_1 > 0$ and $c_2 > 0$

In this subsection, we study the asymptotic properties of $\Re_0$ under the advective effects. To summarize, the main results are as follows:

Theorem 3.1. (Asymptotic profiles of $\Re_0$) *Assume that (A1)-(A2) hold. Then $\Re_0$ satisfies the following properties:*

- **(I)** *For any fixed $D_I > 0$ and $d_I > 0$, $\Re_0 \to \widetilde{\Re}_0$ as $c_1 \to 0$ and $c_2 \to 0$;*
- **(II)** *For any fixed $D_I > 0$ and $d_I > 0$, $\Re_0 \to \Re_0^{\mathrm{loc}}(L)$ as $c_1 \to \infty$ and $c_2 \to \infty$;*
- **(III)** *For any fixed $c_1 > 0$ and $c_2 > 0$, $\Re_0 \to \Re_0^{\mathrm{loc}}(L)$ as $D_I \to 0$ and $d_I \to 0$;*
- **(IV)** *For any fixed $c_1 > 0$ and $c_2 > 0$, $\Re_0 \to \Re_{0a}^{\mathrm{loc}}$ as $D_I \to \infty$ and $d_I \to \infty$;*
- **(V)** *$\Re_0 \to \Re_0^{\mathrm{loc}}(L)$ as $c_1 \to 0$, $c_2 \to 0$, $c_1^2/D_I \to \infty$ and $c_2^2/d_I \to \infty$.*

Before proving Theorem 3.1, the following conclusions are needed.

Lemma 3.2. *If (A1)-(A2) hold, then $\Re_0 \to \Re_0^{\mathrm{loc}}(L)$ as $\sigma \to \infty$, $c_1^2/D_I \to \infty$ and $c_2^2/d_I \to \infty$.*

Proof. Let $(\psi_2, \psi_4) = e^{J\sigma x}(q_2, q_4)$ in $(0, L)$, where $\sigma = c_1/D_I = c_2/d_I$ and (ψ_2, ψ_4) is the positive eigenfunction corresponding to $1/\Re_0$ of (2.6), and J represents some constant which will be selected later for different aims. Since (ψ_2, ψ_4) satisfies (2.6), it follows that

$$\begin{cases} -D_I q_{2xx} + c_1(1-2J)q_{2x} + [c_1\sigma J(1-J) + \alpha_h(x)]q_2 = \dfrac{1}{\Re_0}\hat{\beta}_1(x)q_4, & x\in(0,L), \\ -d_I q_{4xx} + c_2(1-2J)q_{4x} + [c_2\sigma J(1-J) + \mu_v(x)]q_4 = \dfrac{1}{\Re_0}\hat{\beta}_2(x)q_2, & x\in(0,L), \\ D_I q_{2x}(0) = c_1(1-J)q_2(0),\ D_I q_{2x}(L) = c_1(1-J)q_2(L), \\ d_I q_{4x}(0) = c_2(1-J)q_4(0),\ d_I q_{4x}(L) = c_2(1-J)q_4(L). \end{cases} \tag{3.1}$$

First, we set $J = 1 + K_1 D_I/c_1^2$ where $K_1 > 0$ will be chosen later. Then, by (3.1), we have

$$\begin{cases} -D_I q_{2xx} - c_1\left(1+\frac{2K_1D_I}{c_1^2}\right)q_{2x} + \left[-K_1\left(1+\frac{K_1D_I}{c_1^2}\right)+\alpha_h(x)\right]q_2 \\ = \frac{1}{\mathfrak{R}_0}\hat{\beta}_1(x)q_4, \quad x \in (0, L), \\ q_{2x}(0) = -\frac{K_1}{c_1}q_2(0), \ q_{2x}(L) = -\frac{K_1}{c_1}q_2(L). \end{cases} \tag{3.2}$$

Suppose that $q_2(x)$ reaches the minimum value at $x_2^* \in [0, L]$, that is, $q_2(x_2^*) = \min_{x\in[0,L]} q_2(x)$. Due to the positivity of q_2, then $q_{2x}(0) < 0$ from the boundary condition of (3.2) which leads to $x_2^* > 0$. If $x_2^* \in (0, L)$, then $q_{2x}(x_2^*) = 0$ and $q_{2xx}(x_2^*) \geq 0$. Hence, following from (3.2) that

$$\left[-K_1\left(1+\frac{K_1D_I}{c_1^2}\right)+\alpha_h(x_2^*)\right] - \frac{1}{\mathfrak{R}_0}\hat{\beta}_1(x_2^*)\frac{q_4(x_2^*)}{q_2(x_2^*)} \geq 0.$$

If we choose $K_1 = \alpha_h^+$ for any sufficiently small D_I/c_1^2, then

$$\left[-K_1\left(1+\frac{K_1D_I}{c_1^2}\right)+\alpha_h(x_2^*)\right] - \frac{1}{\mathfrak{R}_0}\hat{\beta}_1(x_2^*)\frac{q_4(x_2^*)}{q_2(x_2^*)} < 0$$

which is a contradiction and so $x_2^* = L$. Thus, $q_2(L) \leq q_2(x)$, for any $x \in [0, L]$. By $q_2(x) = e^{-J\sigma x}\psi_2(x)$ and $J = 1 + K_1D_I/c_1^2$, we obtain

$$q_2(L) = e^{-\frac{c_1}{D_I}\left(1+K_1\frac{D_I}{c_1^2}\right)L}\psi_2(L) \leq q_2(x) = e^{-\frac{c_1}{D_I}\left(1+K_1\frac{D_I}{c_1^2}\right)x}\psi_2(x),$$

that is,

$$e^{-\frac{c_1}{D_I}\left(1+K_1\frac{D_I}{c_1^2}\right)(L-x)} \leq \frac{\psi_2(x)}{\psi_2(L)}, \quad \text{for all } x \in [0, L]. \tag{3.3}$$

Moreover, if letting $J = 1 + K_2 d_I/c_2^2$, where $K_2 = \mu_v^+$, then one can similarly obtain

$$e^{-\frac{c_2}{d_I}\left(1+K_2\frac{d_I}{c_2^2}\right)(L-x)} \leq \frac{\psi_4(x)}{\psi_4(L)}, \quad \text{for all } x \in [0, L]. \tag{3.4}$$

Next, choose $J = 1 - K_3 D_I/c_1^2$ where $K_3 > 0$ will be selected later. From (3.1), one gets

$$\begin{cases} -D_I q_{2xx} - c_1\left(1 - \dfrac{2K_3 D_I}{c_1^2}\right) q_{2x} + \left[K_3\left(1 - \dfrac{K_3 D_I}{c_1^2}\right) + \alpha_h(x)\right] q_2 \\ = \dfrac{1}{\mathfrak{R}_0}\hat{\beta}_1(x) q_4, \quad x \in (0, L), \\ q_{2x}(0) = \dfrac{K_3}{c_1} q_2(0), \quad q_{2x}(L) = \dfrac{K_3}{c_1} q_2(L). \end{cases} \tag{3.5}$$

Assume that $q_2(x)$ reaches the maximum value at $x_2^{**} \in [0, L]$, i.e., $q_2(x_2^{**}) = \max_{x\in[0,L]} q_2(x)$. Owing to $q_2(\cdot) > 0$, we have $q_{2x}(0) > 0$ by the boundary condition of (3.5) and so $x_2^{**} > 0$. If $x_2^{**} \in (0, L)$, then $q_{2x}(x_2^{**}) = 0$ and $q_{2xx}(x_2^{**}) \le 0$. Thus, by (3.5), one has

$$\left[K_3\left(1 - \frac{K_3 D_I}{c_1^2}\right) + \alpha_h(x_2^{**})\right] - \frac{1}{\mathfrak{R}_0}\hat{\beta}_1(x_2^{**})\frac{q_4(x_2^{**})}{q_2(x_2^{**})} \le 0. \tag{3.6}$$

Since $\hat{\beta}_1(\cdot)$, $q_2(\cdot)$ and $q_4(\cdot)$ are positive continuous functions on $[0, L]$, there exists a constant $N_1 > 0$ such that $\hat{\beta}_1(x_2^{**}) q_4(x_2^{**})/q_2(x_2^{**}) \le N_1$. Then we choose

$$K_3 = 2N_1\sqrt{\frac{\alpha_h^+\mu_v^+}{\hat{\beta}_1^-\hat{\beta}_2^-}}, \quad \text{and } \frac{D_I}{c_1^2} \text{ is small enough fulfilling } \frac{D_I}{c_1^2} < \frac{1}{4N_1}\sqrt{\frac{\hat{\beta}_1^-\hat{\beta}_2^-}{\alpha_h^+\mu_v^+}} = \frac{1}{2K_3}.$$

This, together with Lemma 2.2, can be concluded that

$$\begin{aligned} & K_3\left(1 - \frac{K_3 D_I}{c_1^2}\right) + \alpha_h(x_2^{**}) - \frac{1}{\mathfrak{R}_0}\hat{\beta}_1(x_2^{**})\frac{q_4(x_2^{**})}{q_2(x_2^{**})} \\ \ge & K_3\left(1 - \frac{K_3 D_I}{c_1^2}\right) + \alpha_h(x_2^{**}) - N_1\sqrt{\frac{\alpha_h^+\mu_v^+}{\hat{\beta}_1^-\hat{\beta}_2^-}} \\ \ge & 2N_1\sqrt{\frac{\alpha_h^+\mu_v^+}{\hat{\beta}_1^-\hat{\beta}_2^-}}\left(1 - 2N_1\sqrt{\frac{\alpha_h^+\mu_v^+}{\hat{\beta}_1^-\hat{\beta}_2^-}}\cdot\frac{1}{4N_1}\sqrt{\frac{\hat{\beta}_1^-\hat{\beta}_2^-}{\alpha_h^+\mu_v^+}}\right) + \alpha_h(x_2^{**}) - N_1\sqrt{\frac{\alpha_h^+\mu_v^+}{\hat{\beta}_1^-\hat{\beta}_2^-}} \\ = & \alpha_h(x_2^{**}) > 0, \end{aligned}$$

which contradicts (3.6) and thus $x_2^{**} = L$. Then $q_2(x) \le q_2(L)$ for all $x \in [0, L]$. By $q_2(x) = e^{-J\sigma x}\psi_2(x)$ and $J = 1 - K_3 D_I/c_1^2$, we obtain

$$q_2(x) = e^{-\frac{c_1}{D_I}\left(1 - K_3\frac{D_I}{c_1^2}\right)x}\psi_2(x) \le q_2(L) = e^{-\frac{c_1}{D_I}\left(1 - K_3\frac{D_I}{c_1^2}\right)L}\psi_2(L),$$

equivalently,

$$\frac{\psi_2(x)}{\psi_2(L)} \le e^{-\frac{c_1}{D_I}\left(1-K_3\frac{D_I}{c_1^2}\right)(L-x)}, \quad \text{for all } x \in [0, L]. \tag{3.7}$$

In the similar fashion, let $J = 1 - K_4 d_I / c_2^2$, where

$$K_4 = 2N_2\sqrt{\frac{\alpha_h^+\mu_v^+}{\hat{\beta}_1^-\hat{\beta}_2^-}}, \text{ and } \frac{d_I}{c_2^2} \text{ is sufficiently small to satisfies } \frac{d_I}{c_2^2} < \frac{1}{4N_2}\sqrt{\frac{\hat{\beta}_1^-\hat{\beta}_2^-}{\alpha_h^+\mu_v^+}} = \frac{1}{2K_4},$$

for some constants $N_2 > 0$. Hence,

$$\frac{\psi_4(x)}{\psi_4(L)} \le e^{-\frac{c_2}{d_I}\left(1-K_4\frac{d_I}{c_2^2}\right)(L-x)}, \text{ for all } x \in [0, L]. \tag{3.8}$$

Consequently, from (3.3), (3.7) and (3.4), (3.8), we obtain

$$e^{-\frac{c_1}{D_I}\left(1+K_1\frac{D_I}{c_1^2}\right)(L-x)} \le \frac{\psi_2(x)}{\psi_2(L)} \le e^{-\frac{c_1}{D_I}\left(1-K_3\frac{D_I}{c_1^2}\right)(L-x)} \tag{3.9}$$

and

$$e^{-\frac{c_2}{d_I}\left(1+K_2\frac{d_I}{c_2^2}\right)(L-x)} \le \frac{\psi_4(x)}{\psi_4(L)} \le e^{-\frac{c_2}{d_I}\left(1-K_4\frac{d_I}{c_2^2}\right)(L-x)}, \quad x \in [0, L]. \tag{3.10}$$

Denote $\xi = c_1(L-x)/D_I = c_2(L-x)/d_I$. By (3.9) and (3.10), one gets

$$e^{-\left(1+K_1\frac{D_I}{c_1^2}\right)\xi} \le \frac{\psi_2\left(L - \frac{D_I}{c_1}\xi\right)}{\psi_2(L)} \le e^{-\left(1-K_3\frac{D_I}{c_1^2}\right)\xi} \tag{3.11}$$

and

$$e^{-\left(1+K_2\frac{d_I}{c_2^2}\right)\xi} \le \frac{\psi_4\left(L - \frac{d_I}{c_2}\xi\right)}{\psi_4(L)} \le e^{-\left(1-K_4\frac{d_I}{c_2^2}\right)\xi}, \quad \xi \in [0, \sigma L]. \tag{3.12}$$

Integrating (2.6) in $(0, L)$ and then dividing by $\psi_2(L)$ and $\psi_4(L)$ respectively, we have

$$\begin{cases} \int_0^L \alpha_h(x)\frac{\psi_2(x)}{\psi_2(L)}\mathrm{d}x = \frac{1}{\Re_0}\int_0^L \hat{\beta}_1(x)\frac{\psi_4(x)}{\psi_2(L)}\mathrm{d}x, \\ \int_0^L \mu_v(x)\frac{\psi_4(x)}{\psi_4(L)}\mathrm{d}x = \frac{1}{\Re_0}\int_0^L \hat{\beta}_2(x)\frac{\psi_2(x)}{\psi_4(L)}\mathrm{d}x. \end{cases}$$

Therefore,

$$\int_0^L \alpha_h(x)\frac{\psi_2(x)}{\psi_2(L)}\mathrm{d}x\int_0^L \mu_v(x)\frac{\psi_4(x)}{\psi_4(L)}\mathrm{d}x = \frac{1}{\mathfrak{R}_0^2}\int_0^L \hat{\beta}_1(x)\frac{\psi_4(x)}{\psi_4(L)}\mathrm{d}x\int_0^L \hat{\beta}_2(x)\frac{\psi_2(x)}{\psi_2(L)}\mathrm{d}x.$$

Since $x = L - D_I\xi/c_1 = L - d_I\xi/c_2$, it follows that

$$\begin{aligned}&\int_0^{\sigma L}\alpha_h\left(L-\frac{D_I}{c_1}\xi\right)\frac{\psi_2\left(L-\frac{D_I}{c_1}\xi\right)}{\psi_2(L)}\mathrm{d}\xi\int_0^{\sigma L}\mu_v\left(L-\frac{d_I}{c_2}\xi\right)\frac{\psi_4\left(L-\frac{d_I}{c_2}\xi\right)}{\psi_4(L)}\mathrm{d}\xi\\ =&\frac{1}{\mathfrak{R}_0^2}\int_0^{\sigma L}\hat{\beta}_1\left(L-\frac{d_I}{c_2}\xi\right)\frac{\psi_4\left(L-\frac{d_I}{c_2}\xi\right)}{\psi_4(L)}\mathrm{d}\xi\int_0^{\sigma L}\hat{\beta}_2\left(L-\frac{D_I}{c_1}\xi\right)\frac{\psi_2\left(L-\frac{D_I}{c_1}\xi\right)}{\psi_2(L)}\mathrm{d}\xi.\end{aligned}$$

Thus, $\mathfrak{R}_0 = \sqrt{G_1(D_I, d_I, c_1, c_2)/G_2(D_I, d_I, c_1, c_2)}$ where

$$G_1(D_I, d_I, c_1, c_2) = \int_0^{\sigma L}\hat{\beta}_1\left(L-\frac{d_I}{c_2}\xi\right)\frac{\psi_4\left(L-\frac{d_I}{c_2}\xi\right)}{\psi_4(L)}\mathrm{d}\xi\int_0^{\sigma L}\hat{\beta}_2\left(L-\frac{D_I}{c_1}\xi\right)\frac{\psi_2\left(L-\frac{D_I}{c_1}\xi\right)}{\psi_2(L)}\mathrm{d}\xi,$$

and

$$G_2(D_I, d_I, c_1, c_2) = \int_0^{\sigma L}\alpha_h\left(L-\frac{D_I}{c_1}\xi\right)\frac{\psi_2\left(L-\frac{D_I}{c_1}\xi\right)}{\psi_2(L)}\mathrm{d}\xi\int_0^{\sigma L}\mu_v\left(L-\frac{d_I}{c_2}\xi\right)\frac{\psi_4\left(L-\frac{d_I}{c_2}\xi\right)}{\psi_4(L)}\mathrm{d}\xi.$$

Combining (3.11)-(3.12) and applying the Lebesgue dominant convergence theorem to yield

$$\begin{aligned}&\lim_{\substack{c_1/D_I\to\infty,\, c_1^2/D_I\to\infty\\ c_2/d_I\to\infty,\, c_2^2/d_I\to\infty}}\mathfrak{R}_0 = \lim_{\substack{c_1/D_I\to\infty,\, c_1^2/D_I\to\infty\\ c_2/d_I\to\infty,\, c_2^2/d_I\to\infty}}\sqrt{\frac{G_1(D_I, d_I, c_1, c_2)}{G_2(D_I, d_I, c_1, c_2)}}\\ =&\lim_{\substack{c_1/D_I\to\infty,\, c_1^2/D_I\to\infty\\ c_2/d_I\to\infty,\, c_2^2/d_I\to\infty}}\sqrt{\frac{\int_0^{\sigma L}\hat{\beta}_1\left(L-\frac{d_I}{c_2}\xi\right)\frac{\psi_4\left(L-\frac{d_I}{c_2}\xi\right)}{\psi_4(L)}\mathrm{d}\xi\int_0^{\sigma L}\hat{\beta}_2\left(L-\frac{D_I}{c_1}\xi\right)\frac{\psi_2\left(L-\frac{D_I}{c_1}\xi\right)}{\psi_2(L)}\mathrm{d}\xi}{\int_0^{\sigma L}\alpha_h\left(L-\frac{D_I}{c_1}\xi\right)\frac{\psi_2\left(L-\frac{D_I}{c_1}\xi\right)}{\psi_2(L)}\mathrm{d}\xi\int_0^{\sigma L}\mu_v\left(L-\frac{d_I}{c_2}\xi\right)\frac{\psi_4\left(L-\frac{d_I}{c_2}\xi\right)}{\psi_4(L)}\mathrm{d}\xi}}\\ =&\sqrt{\frac{\int_0^\infty\hat{\beta}_1(L)e^{-\xi}\mathrm{d}\xi\int_0^\infty\hat{\beta}_2(L)e^{-\xi}\mathrm{d}\xi}{\int_0^\infty\alpha_h(L)e^{-\xi}\mathrm{d}\xi\int_0^\infty\mu_v(L)e^{-\xi}\mathrm{d}\xi}} = \sqrt{\frac{\hat{\beta}_1(L)\hat{\beta}_2(L)}{\alpha_h(L)\mu_v(L)}} = \mathfrak{R}_0^{\mathrm{loc}}(L).\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Assume that (A1)-(A2) hold. For any fixed $c_1 > 0$ and $c_2 > 0$, $\mathfrak{R}_0 \to \mathfrak{R}_{0a}^{\mathrm{loc}}$ as $D_I \to \infty$ and $d_I \to \infty$.*

Proof. According to the uniform boundedness of $\mathfrak{R}_0$ in terms of D_I and d_I by Lemma 2.2, passing to a sequence if necessary, there exists a finite constant $\mathfrak{R}_0^* > 0$ such that $\mathfrak{R}_0 \to \mathfrak{R}_0^*$ when $D_I \to \infty$ and $d_I \to \infty$. Let (ψ_2, ψ_4) be the positive eigenfunction corresponding to $1/\mathfrak{R}_0$ of (2.6). Without loss of generality, assuming $\|\psi_2\| + \|\psi_4\| = 1$. With the help of L^p estimate and the ideas of [27], we know that $\|\psi_2\|_{W^{2,p}((0,L))}$ and $\|\psi_4\|_{W^{2,p}((0,L))}$ are uniformly bounded, for any integer $p > 1$. Then, thanks to the Sobolev embedding theorem [12], $\|\psi_2\|_{C^{1,\alpha}((0,L))}$ and $\|\psi_4\|_{C^{1,\alpha}((0,L))}$ are uniformly bounded, for any $0 \le \alpha \le 1$. Accordingly, there is $\overline{\psi}_j > 0$ such that $\psi_j(\cdot)$ converges to $\overline{\psi}_j$ in $C^1([0, L])$, $j = 2, 4$, as $D_I \to \infty$ and $d_I \to \infty$. Applying the elliptic regularity estimate [12] to yield $\overline{\psi}_2$ and $\overline{\psi}_4$ are constants. Thus, integrating (2.6) over $(0, L)$ and passing to the limit $D_I \to \infty$ and $d_I \to \infty$ to obtain

$$\begin{cases} \overline{\psi}_2 \int_0^L \alpha_h(x)\mathrm{d}x = \overline{\psi}_4 \frac{1}{\mathfrak{R}_0^*} \int_0^L \hat{\beta}_1(x)\mathrm{d}x, \\ \overline{\psi}_4 \int_0^L \mu_v(x)\mathrm{d}x = \overline{\psi}_2 \frac{1}{\mathfrak{R}_0^*} \int_0^L \hat{\beta}_2(x)\mathrm{d}x, \end{cases}$$

which leads to $\mathfrak{R}_0^* = \sqrt{\mathfrak{R}_{0a}^{vh}\mathfrak{R}_{0a}^{hv}} = \mathfrak{R}_{0a}^{\mathrm{loc}}$. □

Proof of Theorem 3.1. (I) is obvious. (II)-(III) and (IV) are the direct consequences of Lemmas 3.2 and 3.3 respectively. For (V), it is straightforward that $c_1/D_I = (c_1^2/D_I)/c_1 \to \infty$ and $c_2/d_I = (c_2^2/d_I)/c_2 \to \infty$ as $c_1^2/D_I \to \infty$, $c_1 \to 0$ and $c_2^2/d_I \to \infty$, $c_2 \to 0$. Then (V) follows from Lemma 3.2. This ends the proof. □

4. Classification of level set of basic reproduction ratio

With the aid of the conclusions obtained in Section 3, in this section, we study the level set classification of $\mathfrak{R}_0$ w.r.t. diffusion rates (D_I, d_I) and advection rates (c_1, c_2), which determines the dynamical behaviors of system (1.1).

Consider the matrix

$$M(x) := e^{\sigma x} \begin{pmatrix} -\alpha_h(x) & \hat{\beta}_1(x) \\ \hat{\beta}_2(x) & -\mu_v(x) \end{pmatrix}.$$

By simple calculations, the principle eigenvalue of $M(x)$ at $x \in (0, L)$ is

$$\kappa_M(x) = e^{\sigma x} \left\{ \frac{-[\alpha_h(x) + \mu_v(x)] + \sqrt{[\alpha_h(x) + \mu_v(x)]^2 + 4[\hat{\beta}_1(x)\hat{\beta}_2(x) - \alpha_h(x)\mu_v(x)]}}{2} \right\}.$$

Let $(e_M^1(x), e_M^2(x))$ be the positive eigenfunction corresponding to $\kappa_M(x)$ at x, i.e.,

$$M(x) \begin{pmatrix} e_M^1(x) \\ e_M^2(x) \end{pmatrix} = \kappa_M(x) \begin{pmatrix} e_M^1(x) \\ e_M^2(x) \end{pmatrix}, \quad x \in (0, L).$$

Denote (ϕ_2^1, ϕ_4^1) as the positive eigenfunction corresponding to $\mathfrak{R}_0 = 1$ (i.e., $\kappa_0 = 1$) of (2.7). To classify the level set of $\mathfrak{R}_0$, the following lemma is necessary:

Lemma 4.1. *Assume that (A1)-(A3) hold. If there exists some sufficiently smooth positive function $k(x)$ such that $(\phi_2^1(x), \phi_4^1(x)) = k(x)(e_M^1(x), e_M^2(x))$, $x \in (0, L)$, then*

(1) *If (H1) is valid, then $\phi_{2x}^1(x) < 0$ and $\phi_{4x}^1(x) < 0$ for $x \in (0, L)$;*
(2) *If (H2) is valid, then $\phi_{2x}^1(x) > 0$ and $\phi_{4x}^1(x) > 0$ for $x \in (0, L)$.*

Proof. By the assumption (A3) and (H1), there is a $x^* \in (0, L)$ such that $\mathfrak{R}_0^{vh}(x)\mathfrak{R}_0^{hv}(x) > 1$ in $(0, x^*)$ and $\mathfrak{R}_0^{vh}(x)\mathfrak{R}_0^{hv}(x) < 1$ in (x^*, L). Recall that (ϕ_2^1, ϕ_4^1) satisfies

$$\begin{cases} -D_I\phi_{2xx}^1 - c_1\phi_{2x}^1 + \alpha_h(x)\phi_2^1 = \hat{\beta}_1(x)\phi_4^1, & x \in (0, L), \\ -d_I\phi_{4xx}^1 - c_2\phi_{4x}^1 + \mu_v(x)\phi_4^1 = \hat{\beta}_2(x)\phi_2^1, & x \in (0, L), \\ \phi_{2x}^1(0) = \phi_{2x}^1(L) = \phi_{4x}^1(0) = \phi_{4x}^1(L) = 0. \end{cases} \tag{4.1}$$

Multiplying the two equations of (4.1) by e^{c_1x/D_I} and e^{c_2x/d_I}, respectively, we get

$$\begin{pmatrix} -D_I(e^{\frac{c_1}{D_I}x}\phi_{2x}^1)_x \\ -d_I(e^{\frac{c_2}{d_I}x}\phi_{4x}^1)_x \end{pmatrix} = e^{\sigma x}\begin{pmatrix} -\alpha_h(x) & \hat{\beta}_1(x) \\ \hat{\beta}_2(x) & -\mu_v(x) \end{pmatrix}\begin{pmatrix} \phi_2^1 \\ \phi_4^1 \end{pmatrix} = M(x)\begin{pmatrix} \phi_2^1 \\ \phi_4^1 \end{pmatrix}, \quad x \in (0, L).$$

Since $(\phi_2^1, \phi_4^1) = k(x)(e_M^1, e_M^2)$ and $M(x)(e_M^1, e_M^2)^T = \kappa_M(x)(e_M^1, e_M^2)^T$, it follows that

$$\begin{pmatrix} -D_I(e^{\frac{c_1}{D_I}x}\phi_{2x}^1)_x \\ -d_I(e^{\frac{c_2}{d_I}x}\phi_{4x}^1)_x \end{pmatrix} = k(x)\kappa_M(x)\begin{pmatrix} e_M^1 \\ e_M^2 \end{pmatrix} = \kappa_M(x)\begin{pmatrix} \phi_2^1 \\ \phi_4^1 \end{pmatrix}, \quad x \in (0, L).$$

According to the above discussion, $\kappa_M > 0$ in $(0, x^*)$ and $\kappa_M < 0$ in (x^*, L). Therefore,

$$(e^{\frac{c_1}{D_I}x}\phi_{2x}^1)_x \begin{cases} < 0, & x \in (0, x^*), \\ > 0, & x \in (x^*, L), \end{cases} \quad \text{and} \quad (e^{\frac{c_2}{d_I}x}\phi_{4x}^1)_x \begin{cases} < 0, & x \in (0, x^*), \\ > 0, & x \in (x^*, L), \end{cases}$$

which implies that $e^{c_1x/D_I}\phi_{2x}^1$ and $e^{c_2x/d_I}\phi_{4x}^1$ monotonically decrease on $(0, x^*)$ and increase on (x^*, L). Owing to $\phi_{jx}^1(0) = \phi_{jx}^1(L) = 0$, $j = 2, 4$, one has $e^{c_1x/D_I}\phi_{2x}^1 < 0$ and $e^{c_2x/d_I}\phi_{4x}^1 < 0$ in $(0, L)$, and thereby $\phi_{jx}^1 < 0$ in $(0, L)$, $j = 2, 4$. The proof of (2) resembles that of (1). □

Remark 4.1. Although the hypothesis in Lemma 4.1 is somewhat harsh, it is necessary in mathematical techniques to cope with the monotonicity of ϕ_2^1 and ϕ_4^1 in $(0, L)$.

4.1. Classification of $\mathfrak{R}_0$ for $\mathfrak{R}_{0a}^{\mathrm{loc}} > 1$

In this subsection, we investigate the classification of $\mathfrak{R}_0$ in the case of $\mathfrak{R}_{0a}^{\mathrm{loc}} > 1$. The main conclusions are as follows:

Theorem 4.1. (Classification of dynamics for $\mathfrak{R}_{0a}^{\mathrm{loc}} > 1$) *Under the conditions of Lemma 4.1, assume that (A1)-(A3) hold. If (H1) is valid, $\mathfrak{R}_{0a}^{\mathrm{loc}} > 1$, and $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ for any $x \in (0, L)$, then there are unique surfaces*

$$\Pi_1 = \{(c_1, \Upsilon_1(D_I, d_I)) : \mathfrak{R}_0(D_I, d_I, \Upsilon_1(D_I, d_I)) = 1, (D_I, d_I) \in (0, \infty)^2\},$$

and

$$\Pi_2 = \{(c_2, \Upsilon_2(D_I, d_I)) : \mathfrak{R}_0(D_I, d_I, \Upsilon_2(D_I, d_I)) = 1, (D_I, d_I) \in (0, \infty)^2\},$$

in spaces $c_1 - (D_I, d_I)$ and $c_2 - (D_I, d_I)$, respectively, such that, for each $D_I > 0$ and $d_I > 0$, system (1.1) is uniformly persistent and admits at least one EE for any $0 < c_1 < \Upsilon_1(D_I, d_I)$ or $0 < c_2 < \Upsilon_2(D_I, d_I)$, and E_0 is g.a.s. for any $c_1 > \Upsilon_1(D_I, d_I)$ or $c_2 > \Upsilon_2(D_I, d_I)$. Furthermore, $\Upsilon_1(D_I, d_I)$ and $\Upsilon_2(D_I, d_I) : (0, \infty)^2 \to (0, \infty)$ satisfy

$$\lim_{D_I \to 0,\, d_I \to 0} \Upsilon_i(D_I, d_I) = 0, \quad \lim_{D_I \to \infty,\, d_I \to \infty} \frac{\Upsilon_1(D_I, d_I)}{D_I} = \Theta_0^*,$$
$$\lim_{D_I \to \infty,\, d_I \to \infty} \frac{\Upsilon_2(D_I, d_I)}{d_I} = \Theta_0^*,$$

where $i = 1, 2$, and Θ_0^* is a positive solution of the equation $G(\Theta) = 0$, here

$$G(\Theta) = \int_0^L e^{\Theta x} \hat{\beta}_1(x) \mathrm{d}x \int_0^L e^{\Theta x} \hat{\beta}_2(x) \mathrm{d}x - \int_0^L e^{\Theta x} \alpha_h(x) \mathrm{d}x \int_0^L e^{\Theta x} \mu_v(x) \mathrm{d}x, \quad \Theta \in [0, \infty).$$

Remark 4.2. Since $c_1/D_I = c_2/d_I$ from the assumption (A2), it follows that $\Upsilon_2 = d_I \Upsilon_1 / D_I$. To state the results of Theorem 4.1 more clearly, we use graphics to elucidate the interesting phenomena. Define the regions

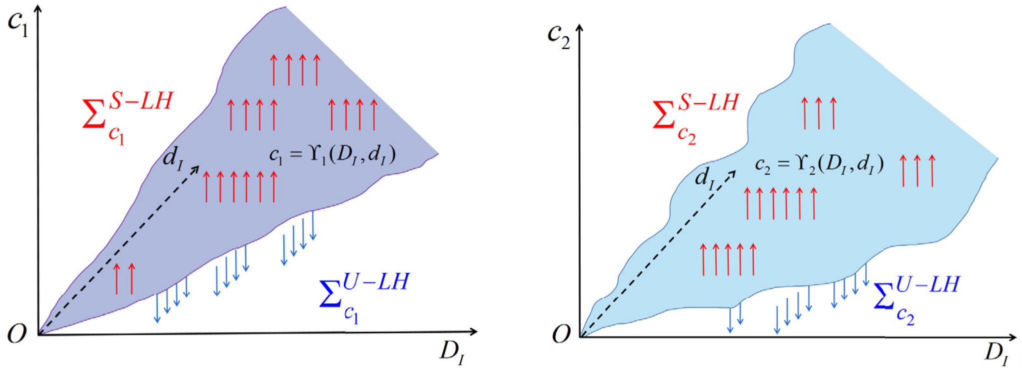
$$\Sigma_{c_i}^{S-LH} = \{(D_I, d_I, c_i) : \mathfrak{R}_0(D_I, d_I, c_i) < 1,\ \mathfrak{R}_0^{\mathrm{loc}}(L) < 1 \text{ and } \mathfrak{R}_{0a}^{\mathrm{loc}} > 1\},$$

and

$$\Sigma_{c_i}^{U-LH} = \{(D_I, d_I, c_i) : \mathfrak{R}_0(D_I, d_I, c_i) > 1,\ \mathfrak{R}_0^{\mathrm{loc}}(L) < 1 \text{ and } \mathfrak{R}_{0a}^{\mathrm{loc}} > 1\},$$

where $i = 1, 2$. The illustrations of dynamic classification in Theorem 4.1 are shown as in Fig. 2.

Furthermore, Theorem 4.1 combining Fig. 2 shows that when the advection rate c_1 (or c_2) are fixed at any value, E_0 is g.a.s. for relatively small diffusion rates D_I and d_I, and system (1.1) is uniformly persistent for relatively large D_I and d_I, which means that the stability of DFE will change at least once as D_I and d_I vary from zero to infinity. Accordingly, the vector-borne disease will die out if c_1/D_I (or c_2/d_I) is large, and will break out if c_1/D_I (or c_2/d_I) is small. From the biological point of view, advection effects convey hosts and vectors to an advantageous place recalling that the downstream end $x = L$ is a low-risk area ($\mathfrak{R}_0^{\mathrm{loc}}(L) < 1$).



(a) (b)

Fig. 2. Description of dynamic classification in Theorem 4.1. The direction of red and blue arrows represent the regions $\Sigma_{c_i}^{S-LH}$ and $\Sigma_{c_i}^{U-LH}$, respectively. In other words, $\Sigma_{c_i}^{S-LH} = \{(D_I, d_I, c_i) : c_i > \Upsilon_i(D_I, d_I), D_I > 0, d_I > 0\}$ and $\Sigma_{c_i}^{U-LH} = \{(D_I, d_I, c_i) : 0 < c_i < \Upsilon_i(D_I, d_I), D_I > 0, d_I > 0\}$. (a) In space $c_1 - (D_I, d_I)$, E_0 is g.a.s. when $(D_I, d_I, c_1) \in \Sigma_{c_1}^{S-LH}$ which suggests that the disease will disappear, and system (1.1) is uniformly persistent when $(D_I, d_I, c_1) \in \Sigma_{c_1}^{U-LH}$ which means that the disease will break out; (b) In space $c_2 - (D_I, d_I)$, E_0 is g.a.s. when $(D_I, d_I, c_2) \in \Sigma_{c_2}^{S-LH}$, and system (1.1) is uniformly persistent when $(D_I, d_I, c_2) \in \Sigma_{c_2}^{U-LH}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The proof of Theorem 4.1 will be completed through one proposition and one lemma.

Proposition 4.1. *Under the conditions of Lemma 4.1, assume that (A1)-(A3) hold. If (H1) is valid,* $\mathfrak{R}_{0a}^{\text{loc}} > 1$, *and* $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ *for any* $x \in (0, L)$, *then*

(1) *For any* $D_I > 0$ *and* $d_I > 0$, *there exists a unique* $c_1^* = c_1^*(D_I, d_I)$ *such that* $\mathfrak{R}_0(D_I, d_I) > 1$ *if* $0 < c_1 < c_1^*$ *and* $\mathfrak{R}_0(D_I, d_I) < 1$ *if* $c_1 > c_1^*$;

(2) *For any* $D_I > 0$ *and* $d_I > 0$, *there exists a unique* $c_2^* = c_2^*(D_I, d_I)$ *such that* $\mathfrak{R}_0(D_I, d_I) > 1$ *if* $0 < c_2 < c_2^*$ *and* $\mathfrak{R}_0(D_I, d_I) < 1$ *if* $c_2 > c_2^*$.

Proof. We just prove the conclusion (1) since the proof of (2) is analogous. Fixed $D_I > 0, d_I > 0$ and $c_2 > 0$. Differentiating the problem (2.7) w.r.t. c_1 to give

$$\begin{cases} -D_I\phi'_{2xx} - \phi_{2x} - c_1\phi'_{2x} + \alpha_h(x)\phi'_2 = -\dfrac{\mathfrak{R}'_0}{\mathfrak{R}_0^2}\hat{\beta}_1(x)\phi_4 + \dfrac{1}{\mathfrak{R}_0}\hat{\beta}_1(x)\phi'_4, & x \in (0, L), \\ -d_I\phi'_{4xx} - c_2\phi'_{4x} + \mu_v(x)\phi'_4 = -\dfrac{\mathfrak{R}'_0}{\mathfrak{R}_0^2}\hat{\beta}_2(x)\phi_2 + \dfrac{1}{\mathfrak{R}_0}\hat{\beta}_2(x)\phi'_2, & x \in (0, L), \\ \phi'_{2x}(0) = \phi'_{2x}(L) = \phi'_{4x}(0) = \phi'_{4x}(L) = 0, \end{cases} \tag{4.2}$$

where $'$ denotes the derivative of c_1. Multiplying the first equation of (2.7) and (4.2) by $e^{c_1 x/D_I}\phi'_2$ and $e^{c_1 x/D_I}\phi_2$ respectively, and then integrating by parts over $(0, L)$ to yield

$$\begin{cases} D_I \int_0^L e^{\frac{c_1}{D_I}x} \phi_{2x}' \phi_{2x} \mathrm{d}x + \int_0^L e^{\frac{c_1}{D_I}x} \alpha_h(x) \phi_2' \phi_2 \mathrm{d}x = \frac{1}{\mathfrak{R}_0} \int_0^L e^{\frac{c_1}{D_I}x} \hat{\beta}_1(x) \phi_2' \phi_4 \mathrm{d}x, \\ D_I \int_0^L e^{\frac{c_1}{D_I}x} \phi_{2x}' \phi_{2x} \mathrm{d}x - \int_0^L e^{\frac{c_1}{D_I}x} \phi_2 \phi_{2x} \mathrm{d}x + \int_0^L e^{\frac{c_1}{D_I}x} \alpha_h(x) \phi_2' \phi_2 \mathrm{d}x \\ = -\frac{\mathfrak{R}_0'}{\mathfrak{R}_0^2} \int_0^L e^{\frac{c_1}{D_I}x} \hat{\beta}_1(x) \phi_2 \phi_4 \mathrm{d}x + \frac{1}{\mathfrak{R}_0} \int_0^L e^{\frac{c_1}{D_I}x} \hat{\beta}_1(x) \phi_2 \phi_4' \mathrm{d}x. \end{cases}$$

Subtracting the above two resulting equations, we get

$$\frac{\mathfrak{R}_0'}{\mathfrak{R}_0^2} \int_0^L e^{\frac{c_1}{D_I}x} \hat{\beta}_1(x) \phi_2 \phi_4 \mathrm{d}x = \int_0^L e^{\frac{c_1}{D_I}x} \phi_2 \phi_{2x} \mathrm{d}x + \frac{1}{\mathfrak{R}_0} \int_0^L e^{\frac{c_1}{D_I}x} \hat{\beta}_1(x) (\phi_2 \phi_4' - \phi_2' \phi_4) \mathrm{d}x. \tag{4.3}$$

In addition, we multiply the second equation of (2.7) and (4.2) by $e^{c_2 x/d_I} \phi_4'$ and $e^{c_2 x/d_I} \phi_4$ respectively, and then integrate by parts over $(0, L)$ to get

$$\begin{cases} d_I \int_0^L e^{\frac{c_2}{d_I}x} \phi_{4x}' \phi_{4x} \mathrm{d}x + \int_0^L e^{\frac{c_2}{d_I}x} \mu_v(x) \phi_4' \phi_4 \mathrm{d}x = \frac{1}{\mathfrak{R}_0} \int_0^L e^{\frac{c_2}{d_I}x} \hat{\beta}_2(x) \phi_4' \phi_2 \mathrm{d}x, \\ d_I \int_0^L e^{\frac{c_2}{d_I}x} \phi_{4x}' \phi_{4x} \mathrm{d}x + \int_0^L e^{\frac{c_2}{d_I}x} \mu_v(x) \phi_4' \phi_4 \mathrm{d}x \\ = -\frac{\mathfrak{R}_0'}{\mathfrak{R}_0^2} \int_0^L e^{\frac{c_2}{d_I}x} \hat{\beta}_2(x) \phi_4 \phi_2 \mathrm{d}x + \frac{1}{\mathfrak{R}_0} \int_0^L e^{\frac{c_2}{d_I}x} \hat{\beta}_2(x) \phi_4 \phi_2' \mathrm{d}x. \end{cases}$$

Hence,

$$\frac{\mathfrak{R}_0'}{\mathfrak{R}_0^2} \int_0^L e^{\frac{c_2}{d_I}x} \hat{\beta}_2(x) \phi_2 \phi_4 \mathrm{d}x = \frac{1}{\mathfrak{R}_0} \int_0^L e^{\frac{c_2}{d_I}x} \hat{\beta}_2(x) (\phi_2' \phi_4 - \phi_2 \phi_4') \mathrm{d}x. \tag{4.4}$$

Adding (4.3) and (4.4), and according to the assumption (A2) and $\hat{\beta}_1 \equiv \hat{\beta}_2$ in $(0, L)$, then

$$\frac{\mathfrak{R}_0'}{\mathfrak{R}_0^2} \int_0^L e^{\sigma x} [\hat{\beta}_1(x) + \hat{\beta}_2(x)] \phi_2 \phi_4 \mathrm{d}x = \int_0^L e^{\sigma x} \phi_2 \phi_{2x} \mathrm{d}x. \tag{4.5}$$

By Lemma 3.2 and the assumption (A2), for fixed $D_I > 0$ and $d_I > 0$, we induce that

$$\lim_{c_1 \to \infty} \mathfrak{R}_0 = \lim_{c_1 \to \infty,\, c_2 \to \infty} \mathfrak{R}_0 = \mathfrak{R}_0^{\mathrm{loc}}(L) < 1$$

which is owing to (H1). Thanks to Theorem 3.1 and Lemma 3.1 (1), one has

$$\lim_{c_1\to 0}\mathfrak{R}_0=\lim_{c_1\to 0,\,c_2\to 0}\mathfrak{R}_0=\widetilde{\mathfrak{R}}_0>1.$$

Therefore, there is at least one point $c_1^*=c_1^*(D_I,d_I)>0$ such that $\mathfrak{R}_0(D_I,d_I,c_1^*,c_2)=1$. To establish the uniqueness of c_1^*, it suffices to prove that $\mathfrak{R}_0'(D_I,d_I,c_1^*,c_2)<0$ for each c_1^* meeting $\mathfrak{R}_0(D_I,d_I,c_1^*,c_2)=1$. According to (4.5) and Lemma 4.1, it follows that

$$\frac{\mathfrak{R}_0'(D_I,d_I,c_1^*,c_2)}{\mathfrak{R}_0^2(D_I,d_I,c_1^*,c_2)}\int_0^L e^{\frac{c_1^*x}{D_I}}[\hat\beta_1(x)+\hat\beta_2(x)]\phi_2\phi_4\mathrm{d}x=\int_0^L e^{\frac{c_1^*x}{D_I}}\phi_2\phi_{2x}\mathrm{d}x<0.$$

By the positivity of $\hat\beta_1$, $\hat\beta_2$ and (ϕ_2,ϕ_4), so $\mathfrak{R}_0'(D_I,d_I,c_1^*,c_2)<0$ which yields that c_1^* is unique. Then $\mathfrak{R}_0(D_I,d_I,c_1,c_2)>1$ for any $0<c_1<c_1^*$ and $\mathfrak{R}_0(D_I,d_I,c_1,c_2)<1$ for any $c_1>c_1^*$. □

Remark 4.3.

(i) It is not conclusive to investigate the monotonicity of $\mathfrak{R}_0$ w.r.t. advection rates, and yet we can find the unique point c_i^* satisfying that $\mathfrak{R}_0(D_I,d_I)$ is large than one if $0<c_i<c_i^*$ and less than one if $c_i>c_i^*$, $i=1,2$.

(ii) By inspection of Proposition 4.1, it is clear that there is unique function $c_i=\Upsilon_i(D_I,d_I)$ of D_I and d_I, such that $\mathfrak{R}_0(D_I,d_I,\Upsilon_1(D_I,d_I),\Upsilon_2(D_I,d_I))=1$, $i=1,2$.

Next, we continue to explore the asymptotic properties of $\Upsilon_i(D_I,d_I)$, $i=1,2$.

Lemma 4.2. *Under the conditions of Proposition 4.1, if (H1) is valid, $\mathfrak{R}_{0a}^{\mathrm{loc}}>1$, and $\hat\beta_1(x)\equiv\hat\beta_2(x)$ for any $x\in(0,L)$, then there exists function $\Upsilon_i(D_I,d_I):(0,\infty)^2\to(0,\infty)$ such that $\mathfrak{R}_0(D_I,d_I,\Upsilon_i(D_I,d_I))=1$, $i=1,2$. Furthermore, $\Upsilon_i(D_I,d_I)$ fulfills*

$$\lim_{D_I\to 0,\,d_I\to 0}\Upsilon_i(D_I,d_I)=0,\quad \lim_{D_I\to\infty,\,d_I\to\infty}\frac{\Upsilon_1(D_I,d_I)}{D_I}=\Theta_0^*,\quad \lim_{D_I\to\infty,\,d_I\to\infty}\frac{\Upsilon_2(D_I,d_I)}{d_I}=\Theta_0^*,$$

where Θ_0^* is a positive solution of the equation $G(\Theta)=0$, where

$$G(\Theta)=\int_0^L e^{\Theta x}\hat\beta_1(x)\mathrm{d}x\int_0^L e^{\Theta x}\hat\beta_2(x)\mathrm{d}x-\int_0^L e^{\Theta x}\alpha_h(x)\mathrm{d}x\int_0^L e^{\Theta x}\mu_v(x)\mathrm{d}x.$$

Proof. Arguing by contradiction, we suppose that there are positive constants $q_0\le\infty$ and $q_1\le\infty$ such that $\Upsilon_1(D_I,d_I)\to q_0$ and $\Upsilon_2(D_I,d_I)\to q_1$ as $D_I\to 0$ and $d_I\to 0$. By Lemma 3.2 and (H1), we have

$$\lim_{\substack{\Upsilon_1(D_I,d_I)\to q_0,\,\Upsilon_1(D_I,d_I)/D_I\to\infty\\ \Upsilon_2(D_I,d_I)\to q_1,\,\Upsilon_2(D_I,d_I)/d_I\to\infty}}\mathfrak{R}_0(D_I,d_I,\Upsilon_1(D_I,d_I),\Upsilon_2(D_I,d_I))=\mathfrak{R}_0^{\mathrm{loc}}(L)<1$$

which contradicts $\mathfrak{R}_0(D_I, d_I, \Upsilon_1(D_I, d_I), \Upsilon_2(D_I, d_I)) = 1$. Consequently, $\Upsilon_i(D_I, d_I) \to 0$ $(i = 1, 2)$ as $D_I \to 0$ and $d_I \to 0$.

If $\Upsilon_1(D_I, d_I)/D_I \to \infty$ and $\Upsilon_2(D_I, d_I)/d_I \to \infty$ as $D_I \to \infty$ and $d_I \to \infty$, then we can obtain a contradiction similarly. Thus, $\Upsilon_1(D_I, d_I)/D_I$ is bounded for sufficiently large D_I. Passing to a subsequence if necessary, we assume that $\Upsilon_1(D_I, d_I)/D_I \to \widehat{\Theta}_1^*$ and $\Upsilon_2(D_I, d_I)/d_I \to \widehat{\Theta}_1^*$ for some nonnegative constant $\widehat{\Theta}_1^*$, as $D_I \to \infty$ and $d_I \to \infty$. Let $(\overline{\phi}_2^*, \overline{\phi}_4^*)$ be the positive eigenfunction corresponding to $\mathfrak{R}_0(D_I, d_I, \Upsilon_1(D_I, d_I), \Upsilon_2(D_I, d_I)) = 1$ of (2.7) satisfying $\|\overline{\phi}_2^*\| + \|\overline{\phi}_4^*\| = 1$. Multiplying the two equations of (2.7) by $e^{\Upsilon_1(D_I,d_I)x/D_I}$ and $e^{\Upsilon_2(D_I,d_I)x/d_I}$ respectively, we have

$$\begin{cases} -D_I\left(e^{\frac{\Upsilon_1(D_I,d_I)}{D_I}x}\overline{\phi}_{2x}^*\right)_x = e^{\frac{\Upsilon_1(D_I,d_I)}{D_I}x}[-\alpha_h(x)\overline{\phi}_2^* + \hat{\beta}_1(x)\overline{\phi}_4^*], & x \in (0, L), \\ -d_I\left(e^{\frac{\Upsilon_2(D_I,d_I)}{d_I}x}\overline{\phi}_{4x}^*\right)_x = e^{\frac{\Upsilon_2(D_I,d_I)}{d_I}x}[-\mu_v(x)\overline{\phi}_4^* + \hat{\beta}_2(x)\overline{\phi}_2^*], & x \in (0, L), \\ \overline{\phi}_{2x}^*(0) = \overline{\phi}_{2x}^*(L) = \overline{\phi}_{4x}^*(0) = \overline{\phi}_{4x}^*(L) = 0. \end{cases}$$

Integrating the above system over $(0, L)$ to give

$$\begin{cases} -\int_0^L e^{\frac{\Upsilon_1(D_I,d_I)}{D_I}x}\alpha_h(x)\overline{\phi}_2^*\mathrm{d}x + \int_0^L e^{\frac{\Upsilon_1(D_I,d_I)}{D_I}x}\hat{\beta}_1(x)\overline{\phi}_4^*\mathrm{d}x = 0, \\ -\int_0^L e^{\frac{\Upsilon_2(D_I,d_I)}{d_I}x}\mu_v(x)\overline{\phi}_4^*\mathrm{d}x + \int_0^L e^{\frac{\Upsilon_2(D_I,d_I)}{d_I}x}\hat{\beta}_2(x)\overline{\phi}_2^*\mathrm{d}x = 0. \end{cases} \tag{4.6}$$

Similar to the analysis of Lemma 3.3, with the aid of the elliptic regularity estimate, there exist positive constants $\widetilde{\phi}_2^*$ and $\widetilde{\phi}_4^*$, such that $(\overline{\phi}_2^*(x), \overline{\phi}_4^*(x)) \to (\widetilde{\phi}_2^*, \widetilde{\phi}_4^*)$ in $(0, L)$ as $D_I \to \infty$ and $d_I \to \infty$. Taking a limit by letting $D_I \to \infty$ and $d_I \to \infty$ in (4.6), one gets

$$\begin{pmatrix} -\int_0^L e^{\widehat{\Theta}_1^* x}\alpha_h(x)\mathrm{d}x & \int_0^L e^{\widehat{\Theta}_1^* x}\hat{\beta}_1(x)\mathrm{d}x \\ \int_0^L e^{\widehat{\Theta}_1^* x}\hat{\beta}_2(x)\mathrm{d}x & -\int_0^L e^{\widehat{\Theta}_1^* x}\mu_v(x)\mathrm{d}x \end{pmatrix}\begin{pmatrix} \widetilde{\phi}_2^* \\ \widetilde{\phi}_4^* \end{pmatrix} := \mathcal{N}\begin{pmatrix} \widetilde{\phi}_2^* \\ \widetilde{\phi}_4^* \end{pmatrix} = 0.$$

Since $\widetilde{\phi}_2^* > 0$ and $\widetilde{\phi}_4^* > 0$, it follows that the matrix $\mathcal{N}$ must be a singular matrix, i.e.,

$$|\mathcal{N}| = \int_0^L e^{\widehat{\Theta}_1^* x}\alpha_h(x)\mathrm{d}x \int_0^L e^{\widehat{\Theta}_1^* x}\mu_v(x)\mathrm{d}x - \int_0^L e^{\widehat{\Theta}_1^* x}\hat{\beta}_1(x)\mathrm{d}x \int_0^L e^{\widehat{\Theta}_1^* x}\hat{\beta}_2(x)\mathrm{d}x = 0.$$

To end the proof, it only remains to show that the equation $G(\Theta) = 0$ admits at least one positive root Θ_0^* by proving the following **Claim**. Actually, $\Theta_0^* = \widehat{\Theta}_1^* > 0$.

Claim. There is some constant $\mathcal{P} > 0$ such that $G(\Theta) < 0$ as $\Theta > \mathcal{P}$.

By utilizing the property of Dirac delta function, we have

$$\begin{aligned}\lim_{\Theta\to\infty}\Theta^2 e^{-2\Theta L}G(\Theta) =& \lim_{\Theta\to\infty}\left[\int_0^L \Theta e^{\Theta(x-L)}\hat{\beta}_1(x)\mathrm{d}x\int_0^L \Theta e^{\Theta(x-L)}\hat{\beta}_2(x)\mathrm{d}x\right] \\ &- \lim_{\Theta\to\infty}\left[\int_0^L \Theta e^{\Theta(x-L)}\alpha_h(x)\mathrm{d}x\int_0^L \Theta e^{\Theta(x-L)}\mu_v(x)\mathrm{d}x\right] \\ =&\hat{\beta}_1(L)\hat{\beta}_2(L)-\alpha_h(L)\mu_v(L)<0\end{aligned}$$

which is due to (H1), and thus $\Theta^2 e^{-2\Theta L}G(\Theta)<0$ for sufficiently large Θ. In other words, there is a constant $\mathcal{P}>0$ large enough such that $\Theta^2 e^{-2\Theta L}G(\Theta)<0$ for $\Theta>\mathcal{P}$, and hence $G(\Theta)<0$ for any $\Theta>\mathcal{P}$. Therefore, the **Claim** is true.

In addition, according to the assumption $\mathfrak{R}_{0a}^{\mathrm{loc}}>1$, we have

$$G(0)=\int_0^L \hat{\beta}_1(x)\mathrm{d}x\int_0^L \hat{\beta}_2(x)\mathrm{d}x-\int_0^L \alpha_h(x)\mathrm{d}x\int_0^L \mu_v(x)\mathrm{d}x>0.$$

By the continuity of $G(\Theta)$ w.r.t. Θ, there is at least $\Theta_0^*>0$ such that $G(\Theta_0^*)=0$. Without loss of generality, we take $\Theta_0^*=\Theta_1^*$. □

Proof of Theorem 4.1. It is obvious to see that Theorem 4.1 is the direct consequences of Proposition 4.1, Lemma 4.2 and Theorem 2.2. □

4.2. Classification of $\mathfrak{R}_0$ for $\mathfrak{R}_{0a}^{\mathrm{loc}}<1$

We subsequently discuss the classification of $\mathfrak{R}_0$ in the case of $\mathfrak{R}_{0a}^{\mathrm{loc}}<1$, and there are the following main conclusions:

Theorem 4.2. (Classification of dynamics for $\mathfrak{R}_{0a}^{\mathrm{loc}}<1$) *Under the conditions of Lemma 4.1, assume that (A1)-(A3) hold, $\mathfrak{R}_{0a}^{\mathrm{loc}}<1$, and $\hat{\beta}_1(x)\equiv\hat{\beta}_2(x)$ for any $x\in(0,L)$. Then there exist two positive constants $\widetilde{D}_I$ and $\widetilde{d}_I$, which is the unique root of the equation $\widetilde{\mathfrak{R}}_0(\widetilde{D}_I,\widetilde{d}_I)=1$, such that the following statements hold:*

(I) *If (H1) is valid, then*
(I-1) *As $(D_I,d_I)\in(0,\widetilde{D}_I)\times(0,\widetilde{d}_I)$, there exist unique surfaces*

$$\Pi_3=\{(c_1,\Upsilon_3(D_I,d_I)):\mathfrak{R}_0(D_I,d_I,\Upsilon_3(D_I,d_I))=1,(D_I,d_I)\in(0,\widetilde{D}_I)\times(0,\widetilde{d}_I)\},$$

and

$$\Pi_4=\{(c_2,\Upsilon_4(D_I,d_I)):\mathfrak{R}_0(D_I,d_I,\Upsilon_4(D_I,d_I))=1,(D_I,d_I)\in(0,\widetilde{D}_I)\times(0,\widetilde{d}_I)\},$$

in spaces $c_1 - (D_I, d_I)$ *and* $c_2 - (D_I, d_I)$*, respectively, such that system* (1.1) *is uniformly persistent and admits at least one EE for any* $0 < c_1 < \Upsilon_3(D_I, d_I)$ *or* $0 < c_2 < \Upsilon_4(D_I, d_I)$*, and* E_0 *is g.a.s. for any* $c_1 > \Upsilon_3(D_I, d_I)$ *or* $c_2 > \Upsilon_4(D_I, d_I)$*. Furthermore,* $\Upsilon_i(D_I, d_I) : (0, \widetilde{D}_I) \times (0, \widetilde{d}_I) \to (0, \infty)$ *fulfills*

$$\lim_{D_I \to 0,\, d_I \to 0} \Upsilon_i(D_I, d_I) = 0, \quad \lim_{D_I \to \widetilde{D}_I^-,\, d_I \to \widetilde{d}_I^-} \Upsilon_i(D_I, d_I) = 0, \quad i = 3, 4;$$

(I-2) *As* $(D_I, d_I) \in [\widetilde{D}_I, \infty) \times [\widetilde{d}_I, \infty)$*,* E_0 *is g.a.s. for any* $c_1 > 0$ *and* $c_2 > 0$*.*

(II) *If (H2) is valid, then*

(II-1) *As* $(D_I, d_I) \in (0, \widetilde{D}_I] \times (0, \widetilde{d}_I]$*, system* (1.1) *is uniformly persistent and admits at least one EE for any* $c_1 > 0$ *and* $c_2 > 0$*;*

(II-2) *As* $(D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)$*, there exist unique surfaces*

$$\Pi_5 = \{(c_1, \Upsilon_5(D_I, d_I)) : \mathfrak{R}_0(D_I, d_I, \Upsilon_5(D_I, d_I)) = 1, (D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)\},$$

and

$$\Pi_6 = \{(c_2, \Upsilon_6(D_I, d_I)) : \mathfrak{R}_0(D_I, d_I, \Upsilon_6(D_I, d_I)) = 1, (D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)\},$$

in spaces $c_1 - (D_I, d_I)$ *and* $c_2 - (D_I, d_I)$*, respectively, such that* E_0 *is g.a.s. for any* $0 < c_1 < \Upsilon_5(D_I, d_I)$ *or* $0 < c_2 < \Upsilon_6(D_I, d_I)$*, and system* (1.1) *is uniformly persistent and admits at least one EE for any* $c_1 > \Upsilon_5(D_I, d_I)$ *or* $c_2 > \Upsilon_6(D_I, d_I)$*. Furthermore,* $\Upsilon_5(D_I, d_I)$ *and* $\Upsilon_6(D_I, d_I) : (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty) \to (0, \infty)$ *are monotonically increasing function of* D_I *and* d_I*, respectively, and fulfill*

$$\lim_{D_I \to \widetilde{D}_I^+,\, d_I \to \widetilde{d}_I^+} \Upsilon_j(D_I, d_I) = 0, \quad \lim_{D_I \to \infty,\, d_I \to \infty} \frac{\Upsilon_5(D_I, d_I)}{D_I} = \Theta_2^*,$$
$$\lim_{D_I \to \infty,\, d_I \to \infty} \frac{\Upsilon_6(D_I, d_I)}{d_I} = \Theta_2^*,$$

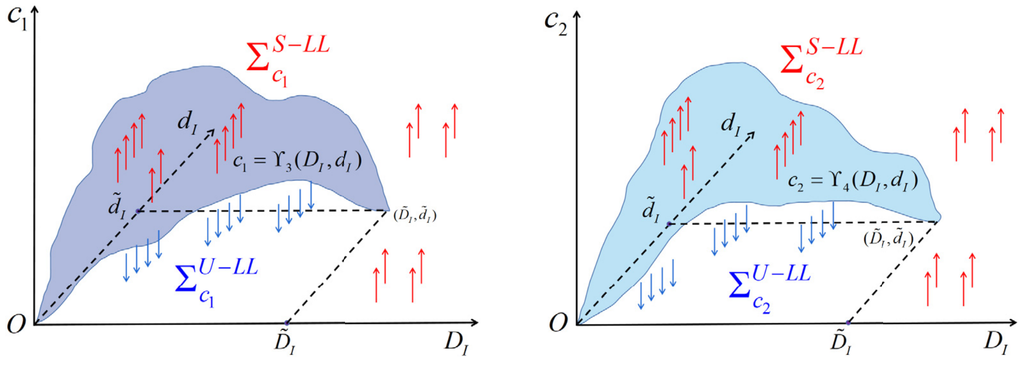
where $j = 5, 6$*, and* Θ_2^* *is a positive solution of the equation* $G(\Theta) = 0$*, here* $G(\Theta)$ *is defined by Theorem 4.1.*

Remark 4.4. Similar to the description of Theorem 4.1 in Remark 4.2, we also utilize graphics to illustrate the meaningful findings of Theorem 4.2. Define the regions

$$\Sigma_{c_i}^{S-LL} = \{(D_I, d_I, c_i) : \mathfrak{R}_0(D_I, d_I, c_i) < 1,\ \mathfrak{R}_0^{\text{loc}}(L) < 1 \text{ and } \mathfrak{R}_{0a}^{\text{loc}} < 1\},$$
$$\Sigma_{c_i}^{U-LL} = \{(D_I, d_I, c_i) : \mathfrak{R}_0(D_I, d_I, c_i) > 1,\ \mathfrak{R}_0^{\text{loc}}(L) < 1 \text{ and } \mathfrak{R}_{0a}^{\text{loc}} < 1\},$$
$$\Sigma_{c_i}^{S-HL} = \{(D_I, d_I, c_i) : \mathfrak{R}_0(D_I, d_I, c_i) < 1,\ \mathfrak{R}_0^{\text{loc}}(L) > 1 \text{ and } \mathfrak{R}_{0a}^{\text{loc}} < 1\},$$

and

$$\Sigma_{c_i}^{U-HL} = \{(D_I, d_I, c_i) : \mathfrak{R}_0(D_I, d_I, c_i) > 1,\ \mathfrak{R}_0^{\text{loc}}(L) > 1 \text{ and } \mathfrak{R}_{0a}^{\text{loc}} < 1\},$$



(a) (b)

Fig. 3. Description of dynamic classification in Theorem 4.2 (I). The direction of red and blue arrows represent the regions $\Sigma_{c_i}^{S-LL}$ and $\Sigma_{c_i}^{U-LL}$, respectively. Namely, $\Sigma_{c_i}^{S-LL} = \Sigma_{c_i}^{S-1-LL} \cup \Sigma_{c_i}^{S-2-LL}$ where $\Sigma_{c_i}^{S-1-LL} = \{(D_I, d_I, c_i) : c_i > \Upsilon_j(D_I, d_I), (D_I, d_I) \in (0, \widetilde{D}_I) \times (0, \widetilde{d}_I)\}$ and $\Sigma_{c_i}^{S-2-LL} = \{(D_I, d_I, c_i) : c_i > 0, (D_I, d_I) \in [\widetilde{D}_I, \infty) \times [\widetilde{d}_I, \infty)\}$, and $\Sigma_{c_i}^{U-LL} = \{(D_I, d_I, c_i) : 0 < c_i < \Upsilon_j(D_I, d_I), (D_I, d_I) \in (0, \widetilde{D}_I) \times (0, \widetilde{d}_I)\}$, $i = 1, 2$, $j = 3, 4$. (a) In space $c_1 - (D_I, d_I)$, E_0 is g.a.s. when $(D_I, d_I, c_1) \in \Sigma_{c_1}^{S-LL}$ which indicates that the disease will die out, and system (1.1) is uniformly persistent when $(D_I, d_I, c_1) \in \Sigma_{c_1}^{U-LL}$ which implies that the disease will break out; (b) In space $c_2 - (D_I, d_I)$, E_0 is g.a.s. when $(D_I, d_I, c_2) \in \Sigma_{c_2}^{S-LH}$, and system (1.1) is uniformly persistent when $(D_I, d_I, c_2) \in \Sigma_{c_2}^{U-LH}$.

where $i = 1, 2$. The illustrations of dynamic classification in Theorem 4.2 are depicted as in Figs. 3–4.

Denote $\overline{\Upsilon}_j := \max\{\Upsilon_j(D_I, d_I) : (D_I, d_I) \in [0, \widetilde{D}_I] \times [0, \widetilde{d}_I]\}$, $j = 3, 4$. According to Fig. 3, Theorem 4.2 (I) shows that: If the advection rate is large enough to make $c_1 > \overline{\Upsilon}_3$ (or $c_2 > \overline{\Upsilon}_4$), no matter what the dispersal of infected hosts and vectors is, the disease will disappear. It should be noted that if $0 < c_1 < \overline{\Upsilon}_3$ (or $0 < c_2 < \overline{\Upsilon}_4$) is fixed, then the stability of E_0 will change at least twice with the increase of D_I and d_I. Namely, when D_I and d_I are sufficiently small or large, E_0 is g.a.s., and when D_I and d_I are between some intermediate values, system (1.1) is uniformly persistent. Biologically, for sufficiently small diffusion rates, advection effects convey hosts and vectors to an advantageous place since the downstream end $x = L$ is a low-risk area ($\mathfrak{R}_0^{\text{loc}}(L) < 1$) and so the disease will die out. For sufficiently large diffusion rates, noticing that the habit $(0, L)$ is a low-risk site ($\mathfrak{R}_{0a}^{\text{loc}} < 1$), the disease will also be eliminated. For intermediate values of diffusion rates, the outbreak or extinction of diseases is neither completely controlled by diffusion nor advection.

From Fig. 4, Theorem 4.2 (II) means that: When $(D_I, d_I) \in (0, \widetilde{D}_I) \times (0, \widetilde{d}_I)$, the disease will persist, regardless of c_1 and c_2. That is, the disease cannot be effectively controlled by reducing the mobility of infected hosts and vectors in this case. When $(D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)$, if c_1/D_I (or c_2/d_I) is relatively small, the disease will die out since the upstream end belongs to a low-risk area ($\mathfrak{R}_0^{\text{loc}}(0) < 1$). If c_1/D_I (or c_2/d_I) is relatively large, the disease will break out since the downstream end belongs to a high-risk area ($\mathfrak{R}_0^{\text{loc}}(L) > 1$).

Combined with the above analysis, the conclusions of Theorem 4.2 indicate that dynamics of (1.1) are not entirely dominated by diffusion rates nor advection rates, but by the combination of the two.

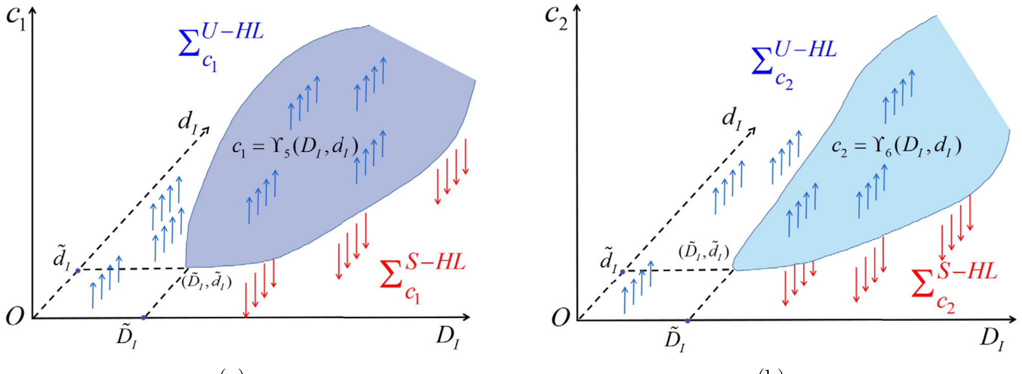


Fig. 4. Description of dynamic classification in Theorem 4.2 (II). The direction of red and blue arrows represent the regions $\Sigma_{c_i}^{S-HL}$ and $\Sigma_{c_i}^{U-HL}$, respectively. That is to say, $\Sigma_{c_i}^{S-HL} = \{(D_I, d_I, c_i) : 0 < c_i < \Upsilon_j(D_I, d_I), (D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)\}$ and $\Sigma_{c_i}^{U-HL} = \Sigma_{c_i}^{U-1-HL} \cup \Sigma_{c_i}^{U-2-HL}$ where $\Sigma_{c_i}^{U-1-HL} = \{(D_I, d_I, c_i) : c_i > \Upsilon_j(D_I, d_I), (D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)\} \cup \{(D_I, d_I, c_i) : c_i > 0, D_I = \widetilde{D}_I, d_I = \widetilde{d}_I\}$ and $\Sigma_{c_i}^{U-2-HL} = \{(D_I, d_I, c_i) : c_i > 0, (D_I, d_I) \in (0, \widetilde{D}_I) \times (0, \widetilde{d}_I)\}$, $i = 1, 2$, $j = 5, 6$. (a) In space $c_1 - (D_I, d_I)$, E_0 is g.a.s. when $(D_I, d_I, c_1) \in \Sigma_{c_1}^{S-HL}$ which yields that the disease will disappear, and system (1.1) is uniformly persistent when $(D_I, d_I, c_1) \in \Sigma_{c_1}^{U-HL}$ which means that the disease will be persistent; (b) In space $c_2 - (D_I, d_I)$, E_0 is g.a.s. when $(D_I, d_I, c_2) \in \Sigma_{c_2}^{S-HL}$, and system (1.1) is uniformly persistent when $(D_I, d_I, c_2) \in \Sigma_{c_2}^{U-HL}$.

Now, we will state some necessary results to prove Theorem 4.2.

Proposition 4.2. *Under the conditions of Lemma 4.1, assume that (A1)-(A3) hold, $\mathfrak{R}_{0a}^{\text{loc}} < 1$, and $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ for any $x \in (0, L)$. Then there exist two positive constants $\widetilde{D}_I$ and $\widetilde{d}_I$, which is the unique root of the equation $\widetilde{\mathfrak{R}}_0(\widetilde{D}_I, \widetilde{d}_I) = 1$, such that the following conclusions hold:*

(1) *If (H1) is valid, then*
 (1-1) *As $(D_I, d_I) \in (0, \widetilde{D}_I) \times (0, \widetilde{d}_I)$, there exist unique points $\tilde{c}_1^* = \tilde{c}_1^*(D_I, d_I)$ and $\tilde{c}_2^* = \tilde{c}_2^*(D_I, d_I)$ such that $\mathfrak{R}_0(D_I, d_I, c_1, c_2) > 1$ for any $0 < c_1 < \tilde{c}_1^*$ or $0 < c_2 < \tilde{c}_2^*$, and $\mathfrak{R}_0(D_I, d_I, c_1, c_2) < 1$ for any $c_1 > \tilde{c}_1^*$ or $c_2 > \tilde{c}_2^*$;*
 (1-2) *As $(D_I, d_I) \in [\widetilde{D}_I, \infty) \times [\widetilde{d}_I, \infty)$, $\mathfrak{R}_0(D_I, d_I, c_1, c_2) < 1$ for any $c_1 > 0$ and $c_2 > 0$.*

(2) *If (H2) is valid, then*
 (2-1) *As $(D_I, d_I) \in (0, \widetilde{D}_I] \times (0, \widetilde{d}_I]$, $\mathfrak{R}_0(D_I, d_I, c_1, c_2) > 1$ for any $c_1 > 0$ and $c_2 > 0$;*
 (2-2) *As $(D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)$, there exist unique points $\hat{c}_1^* = \hat{c}_1^*(D_I, d_I)$ and $\hat{c}_2^* = \hat{c}_2^*(D_I, d_I)$ such that $\mathfrak{R}_0(D_I, d_I, c_1, c_2) < 1$ for any $0 < c_1 < \hat{c}_1^*$ or $0 < c_2 < \hat{c}_2^*$, and $\mathfrak{R}_0(D_I, d_I, c_1, c_2) > 1$ for any $c_1 > \hat{c}_1^*$ or $c_2 > \hat{c}_2^*$.*

Proof. If there exists $\tilde{c}_i^*$ such that $\mathfrak{R}_0(D_I, d_I, \tilde{c}_i^*) = 1$, similar to the proof of Proposition 4.1 and Lemma 3.1 (2), then $\tilde{c}_i^*$ is unique and $\partial \mathfrak{R}_0 / \partial \tilde{c}_i^* < 0$ which suggests that (1-1) is valid. For $D_I \in [\widetilde{D}_I, \infty)$ and $d_I \in [\widetilde{d}_I, \infty)$, it follows from Lemma 3.1 (2) that

$$\lim_{c_1 \to 0,\, c_2 \to 0} \mathfrak{R}_0(c_1, c_2) = \widetilde{\mathfrak{R}}_0 \le 1, \quad \lim_{c_1 \to \infty,\, c_2 \to \infty} \mathfrak{R}_0(c_1, c_2) = \mathfrak{R}_0^{\text{loc}}(L) < 1$$

which indicates that there is no c_i such that $\mathfrak{R}_0(D_I, d_I, c_i) = 1$ and thus $\mathfrak{R}_0(D_I, d_I, c_i) < 1$ for each $c_i > 0$ which implies that (1-2) holds. The proof for the case (H2) is analogous, so we skip the details. □

Remark 4.5. We can see that $c_i = \tilde{c}_i^*(\widetilde{D}_I, \widetilde{d}_I) = 0$, which will be proved below. Specifically, Proposition 4.2 says that if $\mathfrak{R}_0^{\mathrm{loc}}(L) < 1$ (i.e., the downstream end is located in a low-risk area), then $\mathfrak{R}_0(\widetilde{D}_I, \widetilde{d}_I, c_i) < 1$ for each $c_i > \tilde{c}_i^*(\widetilde{D}_I, \widetilde{d}_I) = 0$, and if $\mathfrak{R}_0^{\mathrm{loc}}(L) > 1$ (i.e., the downstream end is located in a high-risk area), then $\mathfrak{R}_0(\widetilde{D}_I, \widetilde{d}_I, c_i) > 1$ for each $c_i > \tilde{c}_i^*(\widetilde{D}_I, \widetilde{d}_I) = 0, i = 1, 2$.

Lemma 4.3. *Under the conditions of Proposition 4.2, assume that (A1)-(A3) hold, $\mathfrak{R}_{0a}^{\mathrm{loc}} < 1$ and $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ for any $x \in (0, L)$. Then there exist two positive constants $\widetilde{D}_I$ and $\widetilde{d}_I$, which is the unique root of the equation $\widetilde{\mathfrak{R}}_0(\widetilde{D}_I, \widetilde{d}_I) = 1$, such that the following conclusions hold:*

(1) *If (H1) is valid, then there exists function $\Upsilon_i(D_I, d_I) : (0, \widetilde{D}_I) \times (0, \widetilde{d}_I) \to (0, \infty)$ such that $\mathfrak{R}_0(D_I, d_I, \Upsilon_i(D_I, d_I)) = 1$. Moreover, $\Upsilon_i(D_I, d_I)$ fulfills*

$$\lim_{D_I \to 0,\, d_I \to 0} \Upsilon_i(D_I, d_I) = 0, \quad \lim_{D_I \to \widetilde{D}_I^-,\, d_I \to \widetilde{d}_I^-} \Upsilon_i(D_I, d_I) = 0, \quad i = 3, 4;$$

(2) *If (H2) is valid, then there exists function $\Upsilon_j(D_I, d_I) : (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty) \to (0, \infty)$ such that $\mathfrak{R}_0(D_I, d_I, \Upsilon_j(D_I, d_I)) = 1$. In addition, $\Upsilon_5(D_I, d_I)$ and $\Upsilon_6(D_I, d_I)$ are monotonically increasing function of D_I and d_I, respectively, and meet*

$$\lim_{D_I \to \widetilde{D}_I^+,\, d_I \to \widetilde{d}_I^+} \Upsilon_j(D_I, d_I) = 0, \quad \lim_{D_I \to \infty,\, d_I \to \infty} \frac{\Upsilon_5(D_I, d_I)}{D_I} = \Theta_2^*,$$
$$\lim_{D_I \to \infty,\, d_I \to \infty} \frac{\Upsilon_6(D_I, d_I)}{d_I} = \Theta_2^*,$$

where $j = 5, 6$, and Θ_2^* is a positive solution of $G(\Theta) = 0$.

Proof. Together with Proposition 4.2, the existence of Υ_i $(i = 3, 4, 5, 6)$ is straightforward.

(1) By utilizing contradicting approach, we suppose that there are $r_3^* > 0$ and $r_4^* > 0$ such that $\Upsilon_3(D_I, d_I) \to r_3^*$ and $\Upsilon_4(D_I, d_I) \to r_4^*$ as $D_I \to 0$ and $d_I \to 0$. Through Lemma 3.2, we have

$$\lim_{\substack{D_I \to 0^+, \Upsilon_3(D_I, d_I) \to r_3^* \\ d_I \to 0^+, \Upsilon_4(D_I, d_I) \to r_4^*}} \mathfrak{R}_0(D_I, d_I, \Upsilon_3(D_I, d_I), \Upsilon_4(D_I, d_I)) = \mathfrak{R}_0^{\mathrm{loc}}(L) < 1$$

which is a contradiction since $\mathfrak{R}_0(D_I, d_I, \Upsilon_3(D_I, d_I), \Upsilon_4(D_I, d_I)) = 1$. Thus, $r_3^* = r_4^* = 0$.

To deal with $\Upsilon_i(D_I, d_I) \to 0$ $(i = 3, 4)$ as $D_I \to \widetilde{D}_I^-$ and $d_I \to \widetilde{d}_I^-$. Assuming there exist $\bar{r}_3^* > 0$ and $\bar{r}_4^* > 0$ such that $\Upsilon_3(D_I, d_I) \to \bar{r}_3^*$ and $\Upsilon_4(D_I, d_I) \to \bar{r}_4^*$ as $D_I \to \widetilde{D}_I^-$ and $d_I \to \widetilde{d}_I^-$. By (2.7), there exists $(\phi_{2\bar{r}_3^*}, \phi_{4\bar{r}_4^*})$ satisfies

$$\begin{cases} -\widetilde{D}_I(\phi_{2\bar{r}_3^*})_{xx} - \bar{r}_3^*(\phi_{2\bar{r}_3^*})_x + \alpha_h(x)\phi_{2\bar{r}_3^*} = \hat{\beta}_1(x)\phi_{4\bar{r}_4^*}, & x \in (0, L), \\ -\widetilde{d}_I(\phi_{4\bar{r}_4^*})_{xx} - \bar{r}_4^*(\phi_{4\bar{r}_4^*})_x + \mu_v(x)\phi_{4\bar{r}_4^*} = \hat{\beta}_2(x)\phi_{2\bar{r}_3^*}, & x \in (0, L), \\ (\phi_{2\bar{r}_3^*})_x(0) = (\phi_{2\bar{r}_3^*})_x(L) = (\phi_{4\bar{r}_4^*})_x(0) = (\phi_{4\bar{r}_4^*})_x(L) = 0. \end{cases} \quad (4.7)$$

Since $\widetilde{\Re}_0(D_I, d_I) = 1$ has a unique root $(\widetilde{D}_I, \widetilde{d}_I)$, there is positive function $(\widetilde{\phi}_{2\bar{r}_3^*}, \widetilde{\phi}_{4\bar{r}_4^*})$ such that

$$\begin{cases} -\widetilde{D}_I(\widetilde{\phi}_{2\bar{r}_3^*})_{xx} + \alpha_h(x)\widetilde{\phi}_{2\bar{r}_3^*} = \hat{\beta}_1(x)\widetilde{\phi}_{4\bar{r}_4^*}, & x \in (0, L), \\ -\widetilde{d}_I(\widetilde{\phi}_{4\bar{r}_4^*})_{xx} + \mu_v(x)\widetilde{\phi}_{4\bar{r}_4^*} = \hat{\beta}_2(x)\widetilde{\phi}_{2\bar{r}_3^*}, & x \in (0, L), \\ (\widetilde{\phi}_{2\bar{r}_3^*})_x(0) = (\widetilde{\phi}_{2\bar{r}_3^*})_x(L) = (\widetilde{\phi}_{4\bar{r}_4^*})_x(0) = (\widetilde{\phi}_{4\bar{r}_4^*})_x(L) = 0. \end{cases} \tag{4.8}$$

Multiplying the two equations of (4.7) by $\widetilde{\phi}_{2\bar{r}_3^*}$ and $\widetilde{\phi}_{4\bar{r}_4^*}$ respectively, and then integrating by parts over $(0, L)$, one obtains

$$\begin{cases} \widetilde{D}_I \int_0^L (\widetilde{\phi}_{2\bar{r}_3^*})_x(\phi_{2\bar{r}_3^*})_x \mathrm{d}x - \bar{r}_3^* \int_0^L \widetilde{\phi}_{2\bar{r}_3^*}(\phi_{2\bar{r}_3^*})_x \mathrm{d}x + \int_0^L \alpha_h(x)\widetilde{\phi}_{2\bar{r}_3^*}\phi_{2\bar{r}_3^*} \mathrm{d}x \\ = \int_0^L \hat{\beta}_1(x)\widetilde{\phi}_{2\bar{r}_3^*}\phi_{4\bar{r}_4^*} \mathrm{d}x, \\ \widetilde{d}_I \int_0^L (\widetilde{\phi}_{4\bar{r}_4^*})_x(\phi_{4\bar{r}_4^*})_x \mathrm{d}x - \bar{r}_4^* \int_0^L \widetilde{\phi}_{4\bar{r}_4^*}(\phi_{4\bar{r}_4^*})_x \mathrm{d}x + \int_0^L \mu_v(x)\widetilde{\phi}_{4\bar{r}_4^*}\phi_{4\bar{r}_4^*} \mathrm{d}x \\ = \int_0^L \hat{\beta}_2(x)\widetilde{\phi}_{4\bar{r}_4^*}\phi_{2\bar{r}_3^*} \mathrm{d}x. \end{cases} \tag{4.9}$$

Similarly, multiplying the two equations of (4.8) by $\phi_{2\bar{r}_3^*}$ and $\phi_{4\bar{r}_4^*}$ respectively, and then integrating by parts over $(0, L)$ to yield

$$\begin{cases} \widetilde{D}_I \int_0^L (\widetilde{\phi}_{2\bar{r}_3^*})_x(\phi_{2\bar{r}_3^*})_x \mathrm{d}x + \int_0^L \alpha_h(x)\widetilde{\phi}_{2\bar{r}_3^*}\phi_{2\bar{r}_3^*} \mathrm{d}x = \int_0^L \hat{\beta}_1(x)\phi_{2\bar{r}_3^*}\widetilde{\phi}_{4\bar{r}_4^*} \mathrm{d}x, \\ \widetilde{d}_I \int_0^L (\widetilde{\phi}_{4\bar{r}_4^*})_x(\phi_{4\bar{r}_4^*})_x \mathrm{d}x + \int_0^L \mu_v(x)\widetilde{\phi}_{4\bar{r}_4^*}\phi_{4\bar{r}_4^*} \mathrm{d}x = \int_0^L \hat{\beta}_2(x)\phi_{4\bar{r}_4^*}\widetilde{\phi}_{2\bar{r}_3^*} \mathrm{d}x. \end{cases} \tag{4.10}$$

Subtracting the two equations of (4.9) and (4.10) and adding the resulting equations, we get

$$\begin{aligned} &- \bar{r}_3^* \int_0^L \widetilde{\phi}_{2\bar{r}_3^*}(\phi_{2\bar{r}_3^*})_x \mathrm{d}x - \bar{r}_4^* \int_0^L \widetilde{\phi}_{4\bar{r}_4^*}(\phi_{4\bar{r}_4^*})_x \mathrm{d}x \\ &= \int_0^L [\hat{\beta}_1(x) - \hat{\beta}_2(x)](\widetilde{\phi}_{2\bar{r}_3^*}\phi_{4\bar{r}_4^*} - \phi_{2\bar{r}_3^*}\widetilde{\phi}_{4\bar{r}_4^*}) \mathrm{d}x = 0 \end{aligned}$$

which is owing to $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ for $x \in (0, L)$. Since $(\phi_{2\bar{r}_3^*})_x < 0$ and $(\phi_{4\bar{r}_3^*})_x < 0$ in $(0, L)$ from Lemma 4.1 (1), we obtain $\bar{r}_3^* = \bar{r}_4^* = 0$. This completes the proof of (1).

(2) Claim that $\partial\mathfrak{R}_0/\partial D_I < 0$ and $\partial\mathfrak{R}_0/\partial d_I < 0$ if D_I and d_I satisfy $\mathfrak{R}_0(D_I, d_I) = 1$. Indeed, differentiating (2.7) w.r.t. D_I to get

$$\begin{cases} -\phi_{2xx} - D_I\phi'_{2xx} - c_1\phi'_{2x} + \alpha_h(x)\phi'_2 = -\dfrac{\mathfrak{R}'_0}{\mathfrak{R}_0^2}\hat{\beta}_1(x)\phi_4 + \dfrac{1}{\mathfrak{R}_0}\hat{\beta}_1(x)\phi'_4, & x \in (0, L), \\ -d_I\phi'_{4xx} - c_2\phi'_{4x} + \mu_v(x)\phi'_4 = -\dfrac{\mathfrak{R}'_0}{\mathfrak{R}_0^2}\hat{\beta}_2(x)\phi_2 + \dfrac{1}{\mathfrak{R}_0}\hat{\beta}_2(x)\phi'_2, & x \in (0, L), \\ \phi'_{2x}(0) = \phi'_{2x}(L) = \phi'_{4x}(0) = \phi'_{4x}(L) = 0, \end{cases} \tag{4.11}$$

wherein $'$ denotes the derivative of D_I. Multiplying the two equations of (2.7) by $e^{c_1x/D_I}\phi'_2$ and $e^{c_2x/d_I}\phi'_4$ respectively, and integrating by parts in $(0, L)$, one has

$$\begin{cases} D_I\displaystyle\int_0^L e^{\frac{c_1}{D_I}x}\phi'_{2x}\phi_{2x}\mathrm{d}x + \int_0^L e^{\frac{c_1}{D_I}x}\alpha_h(x)\phi'_2\phi_2\mathrm{d}x = \frac{1}{\mathfrak{R}_0}\int_0^L e^{\frac{c_1}{D_I}x}\hat{\beta}_1(x)\phi'_2\phi_4\mathrm{d}x, \\ d_I\displaystyle\int_0^L e^{\frac{c_2}{d_I}x}\phi'_{4x}\phi_{4x}\mathrm{d}x + \int_0^L e^{\frac{c_2}{d_I}x}\mu_v(x)\phi'_4\phi_4\mathrm{d}x = \frac{1}{\mathfrak{R}_0}\int_0^L e^{\frac{c_2}{d_I}x}\hat{\beta}_2(x)\phi'_4\phi_2\mathrm{d}x. \end{cases} \tag{4.12}$$

Moreover, we multiply the two equations of (4.11) by $e^{c_1x/D_I}\phi_2$ and $e^{c_2x/d_I}\phi_4$ respectively, and integrate by parts in $(0, L)$ to obtain

$$\begin{cases} \dfrac{c_1}{D_I}\displaystyle\int_0^L e^{\frac{c_1}{D_I}x}\phi_2\phi_{2x}\mathrm{d}x + \int_0^L e^{\frac{c_1}{D_I}x}\phi_{2x}^2\mathrm{d}x + D_I\int_0^L e^{\frac{c_1}{D_I}x}\phi_{2x}\phi'_{2x}\mathrm{d}x \\ +\displaystyle\int_0^L e^{\frac{c_1}{D_I}x}\alpha_h(x)\phi_2\phi'_2\mathrm{d}x = -\frac{\mathfrak{R}'_0}{\mathfrak{R}_0^2}\int_0^L e^{\frac{c_1}{D_I}x}\hat{\beta}_1(x)\phi_2\phi_4\mathrm{d}x + \frac{1}{\mathfrak{R}_0}\int_0^L e^{\frac{c_1}{D_I}x}\hat{\beta}_1(x)\phi_2\phi'_4\mathrm{d}x, \\ d_I\displaystyle\int_0^L e^{\frac{c_2}{d_I}x}\phi_{4x}\phi'_{4x} + \int_0^L e^{\frac{c_2}{d_I}x}\mu_v(x)\phi_4\phi'_4\mathrm{d}x \\ = -\dfrac{\mathfrak{R}'_0}{\mathfrak{R}_0^2}\displaystyle\int_0^L e^{\frac{c_2}{d_I}x}\hat{\beta}_2(x)\phi_4\phi_2\mathrm{d}x + \frac{1}{\mathfrak{R}_0}\int_0^L e^{\frac{c_2}{d_I}x}\hat{\beta}_2(x)\phi_4\phi'_2\mathrm{d}x. \end{cases} \tag{4.13}$$

Subtracting the two equations of (4.12) and (4.13), we get

$$\begin{cases} \dfrac{\mathfrak{R}_0'}{\mathfrak{R}_0^2}\displaystyle\int_0^L e^{\frac{c_1}{D_I}x}\hat{\beta}_1(x)\phi_2\phi_4\mathrm{d}x = -\int_0^L e^{\frac{c_1}{D_I}x}\phi_{2x}^2\mathrm{d}x - \frac{c_1}{D_I}\int_0^L e^{\frac{c_1}{D_I}x}\phi_2\phi_{2x}\mathrm{d}x \\ \qquad\qquad + \dfrac{1}{\mathfrak{R}_0}\displaystyle\int_0^L e^{\frac{c_1}{D_I}x}\hat{\beta}_1(x)(\phi_2\phi_4' - \phi_2'\phi_4)\mathrm{d}x, \\ \dfrac{\mathfrak{R}_0'}{\mathfrak{R}_0^2}\displaystyle\int_0^L e^{\frac{c_2}{d_I}x}\hat{\beta}_2(x)\phi_4\phi_2\mathrm{d}x = \frac{1}{\mathfrak{R}_0}\int_0^L e^{\frac{c_2}{d_I}x}\hat{\beta}_2(x)(\phi_2'\phi_4 - \phi_2\phi_4')\mathrm{d}x. \end{cases}$$

According to the assumption (A2) and $\hat{\beta}_1 \equiv \hat{\beta}_2$ in $(0, L)$, it follows that

$$\frac{\mathfrak{R}_0'}{\mathfrak{R}_0^2}\int_0^L e^{\frac{c_1}{D_I}x}[\hat{\beta}_1(x)+\hat{\beta}_2(x)]\phi_2\phi_4\mathrm{d}x = -\int_0^L e^{\frac{c_1}{D_I}x}\phi_{2x}^2\mathrm{d}x - \frac{c_1}{D_I}\int_0^L e^{\frac{c_1}{D_I}x}\phi_2\phi_{2x}\mathrm{d}x.$$

Applying Lemma 4.1 (2), $\phi_{2x} > 0$ in $(0, L)$ for each $D_I > 0$ and $d_I > 0$ meeting $\mathfrak{R}_0(D_I, d_I) = 1$. Accordingly, $\mathfrak{R}_0' < 0$. Similarly, one show that $\partial\mathfrak{R}_0/\partial d_I < 0$ when $\mathfrak{R}_0(D_I, d_I) = 1$. This proves the claim.

By differentiating the equation $\mathfrak{R}_0(D_I, d_I, \Upsilon_5(D_I, d_I), \Upsilon_6(D_I, d_I)) = 1$ w.r.t. D_I, we get

$$\frac{\partial\mathfrak{R}_0}{\partial D_I} + \frac{\partial\mathfrak{R}_0}{\partial c_1}\cdot\frac{\partial\Upsilon_5(D_I, d_I)}{\partial D_I} + \frac{\partial\mathfrak{R}_0}{\partial c_2}\cdot\frac{\partial\Upsilon_6(D_I, d_I)}{\partial D_I} = 0.$$

Note that $\Upsilon_6(D_I, d_I) = \Upsilon_5(D_I, d_I)d_I/D_I$. Then differentiating it w.r.t. D_I to give

$$\frac{\partial\Upsilon_6(D_I, d_I)}{\partial D_I} = -\frac{d_I}{D_I^2}\cdot\Upsilon_5(D_I, d_I) + \frac{d_I}{D_I}\cdot\frac{\partial\Upsilon_5(D_I, d_I)}{\partial D_I}.$$

Thus,

$$\frac{\partial\Upsilon_5(D_I, d_I)}{\partial D_I}\left(\frac{\partial\mathfrak{R}_0}{\partial c_1} + \frac{d_I}{D_I}\cdot\frac{\partial\mathfrak{R}_0}{\partial c_2}\right) = \frac{d_I}{D_I^2}\cdot\Upsilon_5(D_I, d_I)\cdot\frac{\partial\mathfrak{R}_0}{\partial c_2} - \frac{\partial\mathfrak{R}_0}{\partial D_I}.$$

Since $\partial\mathfrak{R}_0/\partial D_I < 0$ from the above claim and $\partial\mathfrak{R}_0/\partial c_i > 0$ $(i = 1, 2)$ due to Proposition 4.2 (2-2), it can be summarized that $\partial\Upsilon_5/\partial D_I > 0$ which indicates that Υ_5 is a increasing function of D_I. Similarly, $\partial\Upsilon_6/\partial d_I > 0$, that is, Υ_6 is a increasing function of d_I.

Similar to the analysis of (1) and Lemma 4.2, we can show that

$$\lim_{D_I\to\widetilde{D}_I^+,\, d_I\to\widetilde{d}_I^+}\Upsilon_j(D_I, d_I) = 0, \quad \lim_{D_I\to\infty,\, d_I\to\infty}\frac{\Upsilon_5(D_I, d_I)}{D_I} = \Theta_2^*,$$
$$\lim_{D_I\to\infty,\, d_I\to\infty}\frac{\Upsilon_6(D_I, d_I)}{d_I} = \Theta_2^*,$$

wherein $j = 5, 6$ and Θ_2^* is a positive solution of $G(\Theta) = 0$. □

5. Aggregation phenomenon of endemic equilibrium

Recall that system (1.1) has at least an EE if $\mathfrak{R}_0$ is greater than one by Theorem 2.2. Throughout this section, we choose $f_1(\cdot, I_v) = f_{11}(\cdot, I_v)I_v$ and $f_2(\cdot, I_h) = f_{22}(\cdot, I_h)I_h$ satisfying $\partial_{I_v} f_{11}(\cdot, I_v) \le 0$ and $\partial_{I_h} f_{22}(\cdot, I_h) \le 0$ in $(0, L)$. Obviously, $\partial_{I_v} f_1(\cdot, 0) = f_{11}(\cdot, 0)$ and $\partial_{I_h} f_2(\cdot, 0) = f_{22}(\cdot, 0)$. We intend to discuss the aggregation behaviors of EE. To summarize, we have the following main results.

Theorem 5.1. (**Exponential decay**) *Assume that (A1)-(A2) hold. If (H2) is valid, then there exists some constant $C_7 > 0$ such that, for any σ, c_1^2/D_I, $c_2^2/d_I > C_7$, system (1.1) has at least an EE $E_* = (S_h(x), I_h(x), R_h(x), S_v(x), I_v(x))$ such that the following statements hold:*

(I) *If $I_h(L) > I_v(L)$, then there exists a constant $C_8 > 0$ such that*

$$\left| \frac{I_h(x)}{I_h(L)} - e^{-\frac{c_1}{D_I}(L-x)} \right| \le \frac{C_8 D_I}{c_1^2} e^{-\frac{c_1}{2D_I}(L-x)}, \quad for\ all\ x \in [0, L]. \tag{5.1}$$

(II) *If $I_h(L) < I_v(L)$, then there exists a constant $C_9 > 0$ such that*

$$\left| \frac{I_v(x)}{I_v(L)} - e^{-\frac{c_2}{d_I}(L-x)} \right| \le \frac{C_9 d_I}{c_2^2} e^{-\frac{c_2}{2d_I}(L-x)}, \quad for\ all\ x \in [0, L]. \tag{5.2}$$

(III) *If $I_h(L) = I_v(L)$, then (5.1) and (5.2) are both valid.*

Remark 5.1. Biologically, Theorem 5.1 indicates that when c_1/D_I or c_2/d_I is relatively large (i.e., the influence of advection is dominant relative to the dispersal), the infected hosts or vectors will be aggregated at the downstream end $x = L$.

Theorem 5.2. (**Limiting profile**) *Let E_* be the EE of system (1.1). Under the conditions of Theorem 5.1, then the following conclusions hold:*

(I) *If $I_h(L) > I_v(L)$, then*

$$\lim_{c_1/D_I \to \infty,\, c_1^2/D_I \to \infty} \frac{c_1}{D_I I_h(L)} \int_0^L I_h(x)\mathrm{d}x = 1. \tag{5.3}$$

(II) *If $I_h(L) < I_v(L)$, then*

$$\lim_{c_2/d_I \to \infty,\, c_2^2/d_I \to \infty} \frac{c_2}{d_I I_v(L)} \int_0^L I_v(x)\mathrm{d}x = 1. \tag{5.4}$$

(III) *If $I_h(L) = I_v(L)$, then (5.3) and (5.4) are both valid.*

To prove Theorems 5.1 and 5.2, we shall present some necessary preliminaries.

Lemma 5.1. *Suppose that (u, v) is a solution of the elliptic system:*

$$\begin{cases} d_1 u_{1xx} - c_1 u_{1x} + h_1(x)u_2 - r_1(x)u_1 \le 0, & x \in (0, L), \\ d_2 u_{2xx} - c_2 u_{2x} + h_2(x)u_1 - r_2(x)u_2 \le 0, & x \in (0, L), \\ -d_1 u_{1x}(0) + c_1 u_1(0) \ge 0,\ u_1(L) \ge 0, \\ -d_2 u_{2x}(0) + c_2 u_2(0) \ge 0,\ u_2(L) \ge 0, \end{cases} \tag{5.5}$$

where $d_i > 0,\ c_i > 0,\ h_i(\cdot) > 0$ and $h_i(\cdot),\ r_i(\cdot) \in C([0, L]),\ i = 1, 2$. If $c_1/d_1 = c_2/d_2$ and $c_i^2/d_i \ge 4\max\{r_i(x) : x \in [0, L]\}$, then $u_i(x) \equiv 0$ or $u_i(x) > 0$ for any $x \in [0, L)$, $i = 1, 2$.

Proof. Let $(u_1, u_2) = e^{\zeta x/2}(\tilde{u}_1, \tilde{u}_2)$, where $\zeta = c_1/d_1 = c_2/d_2$. Then $(\tilde{u}_1, \tilde{u}_2)$ satisfies

$$\begin{cases} \left(d_1\partial_x^2 - \dfrac{c_1^2}{2d_1}\right)\tilde{u}_1 + h_1(x)\tilde{u}_2 + \left[\dfrac{c_1^2}{4d_1} - r_1(x)\right]\tilde{u}_1 \le 0, & x \in (0, L), \\ \left(d_2\partial_x^2 - \dfrac{c_2^2}{2d_2}\right)\tilde{u}_2 + h_2(x)u_1 + \left[\dfrac{c_2^2}{4d_2} - r_2(x)\right]\tilde{u}_2 \le 0, & x \in (0, L), \\ -d_1\tilde{u}_{1x}(0) + \dfrac{c_1}{2}\tilde{u}_1(0) \ge 0,\ \tilde{u}_1(L) \ge 0, \\ -d_2\tilde{u}_{2x}(0) + \dfrac{c_2}{2}\tilde{u}_2(0) \ge 0,\ \tilde{u}_2(L) \ge 0. \end{cases} \tag{5.6}$$

We can verify that system (5.6) is quasimonotone nondecreasing (or cooperative). Then, applying the strong maximum principle for elliptic equations [29, Lemma 2.4] and [40, Lemma 2.1.2] yields that $\tilde{u}_i(x) \equiv 0$ or $\tilde{u}_i(x) > 0$ for any $x \in [0, L)$, $i = 1, 2$. Therefore, $u_i(x) \equiv 0$ or $u_i(x) > 0$ for any $x \in [0, L)$, $i = 1, 2$. □

Remark 5.2. Lemma 5.1 extends the conclusions containing a single elliptic equation in [5, Lemma 3.1] to those including two elliptic equations.

Lemma 5.2. *Let $E_* = (S_h(x), I_h(x), R_h(x), S_v(x), I_v(x))$ be the EE of system (1.1). Assume that (A1)-(A2) hold, and $c_1^2/D_I > (\delta_1^*)^2$ and $c_2^2/d_I > (\delta_2^*)^2$, where $\delta_1^* = \alpha_h^+ + \hat{\beta}_1^+ + \delta_1^0 + 2$ and $\delta_2^* = \mu_v^+ + \hat{\beta}_2^+ + \delta_2^0 + 2$, here δ_1^0 and δ_1^0 are positive constants such that $\delta_1^*/c_1 = \delta_2^*/c_2$. Then*

$$\underline{I}_h^*(x) \le I_h(x) \le \overline{I}_h^*(x),\ \ \underline{I}_v^*(x) \le I_v(x) \le \overline{I}_v^*(x),\ \ for\ any\ x \in [0, L],$$

where

$$\overline{I}_h^*(x) = K_5 e^{-(\frac{c_1}{D_I} - \frac{\delta_1^*}{c_1})(L-x)},\ \ \underline{I}_h^*(x) = I_h(L)e^{-(\frac{c_1}{D_I} + \frac{\delta_1^*}{c_1})(L-x)},$$

and

$$\overline{I}_v^*(x) = K_5 e^{-(\frac{c_2}{d_I} - \frac{\delta_2^*}{c_2})(L-x)},\ \ \underline{I}_v^*(x) = I_v(L)e^{-(\frac{c_2}{d_I} + \frac{\delta_2^*}{c_2})(L-x)},\ \ x \in [0, L],$$

wherein $K_5 = \max\{I_h(L), I_v(L)\}$.

Proof. To prove $\overline{I}_h^*(x)$ and $\overline{I}_v^*(x)$ are super-solutions. By simple calculations, one gets

$$\begin{aligned}
& D_I \overline{I}_{hxx}^* - c_1 \overline{I}_{hx}^* + \beta_1(x) \frac{S_h f_{11}(x, I_v)}{S_h + I_h + R_h} \overline{I}_v^* - \alpha_h(x) \overline{I}_h^* \\
&= D_I \left(\frac{c_1}{D_I} - \frac{\delta_1^*}{c_1} \right)^2 \overline{I}_h^* - c_1 \left(\frac{c_1}{D_I} - \frac{\delta_1^*}{c_1} \right) \overline{I}_h^* - \alpha_h(x) \overline{I}_h^* + \beta_1(x) \frac{S_h f_{11}(x, I_v)}{S_h + I_h + R_h} \overline{I}_v^* \\
&\le \left[-\delta_1^* + \frac{D_I (\delta_1^*)^2}{c_1^2} - \alpha_h(x) \right] \overline{I}_h^* + \hat{\beta}_1(x) \overline{I}_v^* \\
&\le \left[-\delta_1^* + \frac{D_I (\delta_1^*)^2}{c_1^2} \right] \overline{I}_h^* + \hat{\beta}_1(x) \overline{I}_v^* \\
&= \left[-\delta_1^* + \frac{D_I (\delta_1^*)^2}{c_1^2} \right] K_5 e^{-(\frac{c_1}{D_I} - \frac{\delta_1^*}{c_1})(L-x)} + \hat{\beta}_1(x) K_5 e^{-(\frac{c_2}{d_I} - \frac{\delta_2^*}{c_2})(L-x)} \\
&= \left[-\delta_1^* + \frac{D_I (\delta_1^*)^2}{c_1^2} + \hat{\beta}_1^+ \right] K_5 e^{-(\frac{c_1}{D_I} - \frac{\delta_1^*}{c_1})(L-x)} \\
&\le \left[-1 + \frac{D_I (\delta_1^*)^2}{c_1^2} \right] K_5 e^{-(\frac{c_1}{D_I} - \frac{\delta_1^*}{c_1})(L-x)} \le 0,
\end{aligned}$$

and

$$d_I \overline{I}_{vxx}^* - c_2 \overline{I}_{vx}^* + \beta_2(x) \frac{S_v f_{22}(x, I_h)}{S_v + I_v} \overline{I}_h^* - \mu_v(x) \overline{I}_v^* \le \left[-1 + \frac{d_I (\delta_2^*)^2}{c_2^2} \right] K_5 e^{-(\frac{c_2}{d_I} - \frac{\delta_2^*}{c_2})(L-x)} \le 0,$$

and

$$\begin{cases}
- D_I \overline{I}_{hx}^*(0) + c_1 \overline{I}_h^*(0) = -D_I \left(\frac{c_1}{D_I} - \frac{\delta_1^*}{c_1} \right) \overline{I}_h^*(0) + c_1 \overline{I}_h^*(0) = \frac{D_I \delta_1^*}{c_1} \overline{I}_h^*(0) \ge 0, \\
\overline{I}_h^*(L) = K_5, \\
- d_I \overline{I}_{vx}^*(0) + c_2 \overline{I}_v^*(0) = -d_I \left(\frac{c_2}{d_I} - \frac{\delta_2^*}{c_2} \right) \overline{I}_v^*(0) + c_2 \overline{I}_v^*(0) = \frac{d_I \delta_2^*}{c_2} \overline{I}_v^*(0) \ge 0, \\
\overline{I}_v^*(L) = K_5,
\end{cases}$$

where the assumptions (A2) and $\delta_1^*/c_1 = \delta_2^*/c_2$ are used.

Setting $u_1 = \overline{I}_h^* - I_h$ and $u_2 = \overline{I}_v^* - I_v$, thus we have

$$\begin{aligned}
& D_I u_{1xx} - c_1 u_{1x} + \beta_1(x) \frac{S_h f_{11}(x, I_v)}{S_h + I_h + R_h} u_2 - \alpha_h(x) u_1 \\
=& D_I (\overline{I}_h^* - I_h)_{xx} - c_1 (\overline{I}_h^* - I_h)_x + \beta_1(x) \frac{S_h f_{11}(x, I_v)}{S_h + I_h + R_h} (\overline{I}_v^* - I_v) - \alpha_h(x) (\overline{I}_h^* - I_h)
\end{aligned}$$

$$
\begin{aligned}
=&D_I\overline{I}^*_{hxx} - c_1\overline{I}^*_{hx} + \beta_1(x)\frac{S_h f_{11}(x,I_v)}{S_h+I_h+R_h}\overline{I}^*_v - \alpha_h(x)\overline{I}^*_h \\
&-\left[D_I I_{hxx} - c_1 I_{hx} + \beta_1(x)\frac{S_h f_{11}(x,I_v)}{S_h+I_h+R_h}I_v - \alpha_h(x)I_h\right] \\
=&D_I\overline{I}^*_{hxx} - c_1\overline{I}^*_{hx} + \beta_1(x)\frac{S_h f_{11}(x,I_v)}{S_h+I_h+R_h}\overline{I}^*_v - \alpha_h(x)\overline{I}^*_h \le 0,
\end{aligned}
$$

and

$$
d_I u_{2xx} - c_2 u_{2x} + \beta_2(x)\frac{S_v f_{22}(x,I_h)}{S_v+I_v}u_1 - \mu_v(x)u_2 \le 0,
$$

and

$$
\begin{cases}
-D_I u_{1x}(0) + c_1 u_1(0) = -D_I\overline{I}^*_{hx}(0) + c_1\overline{I}^*_h(0) = \dfrac{D_I\delta_1^*}{c_1}\overline{I}^*_h(0) \ge 0, \\
u_1(L) = \overline{I}^*_h(L) - I_h(L) = K_5 - I_h(L) \ge 0, \\
-d_I u_{2x}(0) + c_2 u_2(0) = -d_I\overline{I}^*_{vx}(0) + c_2\overline{I}^*_v(0) = \dfrac{d_I\delta_2^*}{c_2}\overline{I}^*_v(0) \ge 0, \\
u_2(L) = \overline{I}^*_v(L) - I_v(L) = K_5 - I_v(L) \ge 0.
\end{cases}
$$

By using Lemma 5.1, we see that $u_i(x) \ge 0$, $i = 1, 2$, $x \in [0, L)$, which implies that $I_h(x) \le \overline{I}^*_h(x)$ and $I_v(x) \le \overline{I}^*_v(x)$, $x \in [0, L)$. Similarly, we can testify that $\underline{I}^*_h(x)$ and $\underline{I}^*_v(x)$ are subsolutions, i.e., $I_h(x) \ge \underline{I}^*_h(x)$ and $I_v(x) \ge \underline{I}^*_v(x)$, $x \in [0, L)$. □

Lemma 5.3. *Consider the functions*

$$
F_1^+(\xi) := e^{\frac{\delta_1^*}{c_1}\xi} - \frac{3\delta_1^* D_I}{c_1^2}e^{\frac{c_1}{2D_I}\xi} - 1,\ \ F_1^-(\xi) := e^{-\frac{\delta_1^*}{c_1}\xi} + \frac{3\delta_1^* D_I}{c_1^2}e^{\frac{c_1}{2D_I}\xi} - 1,
$$

and

$$
F_2^+(\xi) := e^{\frac{\delta_2^*}{c_2}\xi} - \frac{3\delta_2^* d_I}{c_2^2}e^{\frac{c_2}{2d_I}\xi} - 1,\ \ F_2^-(\xi) := e^{-\frac{\delta_2^*}{c_2}\xi} + \frac{3\delta_2^* d_I}{c_2^2}e^{\frac{c_2}{2d_I}\xi} - 1,\ \ \xi \in [0, L],
$$

where δ_i^* is determined by Lemma 5.2. Then $F_i^+(\xi) \le 0$ and $F_i^-(\xi) \ge 0$, $i = 1, 2$, $\xi \in [0, L]$.

Proof. By direct calculations, one has

$$
\begin{aligned}
F_1^{+'}(\xi) &= \frac{\delta_1^*}{c_1}e^{\frac{\delta_1^*}{c_1}\xi} - \frac{3\delta_1^*}{2c_1}e^{\frac{c_1}{2D_I}\xi} = \frac{\delta_1^*}{c_1}e^{\frac{\delta_1^*}{c_1}\xi}\left[1 - \frac{3}{2}e^{(\frac{c_1}{2D_I} - \frac{\delta_1^*}{c_1})\xi}\right] \\
&\le \frac{\delta_1^*}{c_1}e^{\frac{\delta_1^*}{c_1}\xi}\left(1 - \frac{3}{2}\right) \le 0,\ \ \xi \in [0, L],
\end{aligned}
$$

which is due to $c_1^2/(2D_I) > (\delta_1^*)^2/2 > \delta_1^*$. Following from $F_1^+(0) < 0$ that $F_1^+(\xi) \le 0$, $\xi \in [0, L]$. Similarly, one can show $F_2^+(\xi) \le 0$ and $F_i^-(\xi) \ge 0$, $\xi \in [0, L]$, $i = 1, 2$. This ends the proof. □

Afterwards, we complete the proof of Theorems 5.1 and 5.2.

Proof of Theorem 5.1. According to Lemma 3.2, there is a sufficiently large constant $C_7 > 0$ such that $\mathfrak{R}_0 > 1$ for $\sigma, c_1^2/D_I, c_2^2/d_I > C_7$. Then, together with Theorem 2.2, system (1.1) has at least an EE E_*.

If $I_h(L) > I_v(L)$, then $K_5 = I_h(L)$ by Lemma 5.2 and

$$I_h(L)e^{-(\frac{c_1}{D_I}+\frac{\delta_1^*}{c_1})(L-x)} \le I_h(x) \le I_h(L)e^{-(\frac{c_1}{D_I}-\frac{\delta_1^*}{c_1})(L-x)}, \;\; x \in [0, L].$$

Subtracting $I_h(L)e^{-c_1(L-x)/D_I}$ from both ends of above inequality to give

$$\begin{aligned} e^{-\frac{c_1}{D_I}(L-x)}\left[e^{-\frac{\delta_1^*}{c_1}(L-x)}-1\right]I_h(L) &\le I_h(x) - I_h(L)e^{-\frac{c_1}{D_I}(L-x)} \\ &\le \left[e^{\frac{\delta_1^*}{c_1}(L-x)}-1\right]e^{-\frac{c_1}{D_I}(L-x)}I_h(L). \end{aligned} \tag{5.7}$$

From Lemma 5.3, letting $\xi = L - x$, then

$$F_1^+(L-x) = e^{\frac{\delta_1^*}{c_1}(L-x)} - \frac{3\delta_1^* D_I}{c_1^2}e^{\frac{c_1}{2D_I}(L-x)} - 1 \le 0,$$

and

$$F_1^-(L-x) = e^{-\frac{\delta_1^*}{c_1}(L-x)} + \frac{3\delta_1^* D_I}{c_1^2}e^{\frac{c_1}{2D_I}(L-x)} - 1 \ge 0, \;\; x \in [0, L].$$

Therefore,

$$e^{\frac{\delta_1^*}{c_1}(L-x)} - 1 \le \frac{3\delta_1^* D_I}{c_1^2}e^{\frac{c_1}{2D_I}(L-x)}, \text{ and } e^{-\frac{\delta_1^*}{c_1}(L-x)} - 1 \ge -\frac{3\delta_1^* D_I}{c_1^2}e^{\frac{c_1}{2D_I}(L-x)}, \;\; x \in [0, L].$$

Substituting it into (5.7) to yield

$$\begin{aligned} &- I_h(L)\frac{3\delta_1^* D_I}{c_1^2}e^{-\frac{c_1}{2D_I}(L-x)} = -I_h(L)\frac{3\delta_1^* D_I}{c_1^2}e^{-\frac{c_1}{D_I}(L-x)}e^{\frac{c_1}{2D_I}(L-x)} \\ &\le \left[e^{-\frac{\delta_1^*}{c_1}(L-x)}-1\right]e^{-\frac{c_1}{D_I}(L-x)}I_h(L) \le I_h(x) - I_h(L)e^{-\frac{c_1}{D_I}(L-x)} \end{aligned}$$

$$\leq \left[e^{\frac{\delta_1^*}{c_1}(L-x)} - 1 \right] e^{-\frac{c_1}{D_I}(L-x)} I_h(L) \leq \frac{3\delta_1^* D_I}{c_1^2} e^{\frac{c_1}{2D_I}(L-x)} e^{-\frac{c_1}{D_I}(L-x)} I_h(L)$$

$$= I_h(L) \frac{3\delta_1^* D_I}{c_1^2} e^{-\frac{c_1}{2D_I}(L-x)}.$$

Hence,

$$\left| \frac{I_h(x)}{I_h(L)} - e^{-\frac{c_1}{D_I}(L-x)} \right| \leq \frac{3\delta_1^* D_I}{c_1^2} e^{-\frac{c_1}{2D_I}(L-x)}.$$

In a similar way, we can cope with (5.2) when $I_h(L) < I_v(L)$. □

Proof of Theorem 5.2. If $I_h(L) > I_v(L)$, letting $\xi = c_1(L-x)/D_I$, it then follows from Lemma 5.2 that

$$I_h(L) e^{-\left(1 + \frac{\delta_1^* D_I}{c_1^2}\right)\xi} \leq I_h\left(L - \frac{D_I}{c_1}\xi\right) \leq I_h(L) e^{-\left(1 - \frac{\delta_1^* D_I}{c_1^2}\right)\xi}, \quad \xi \in [0, c_1 L/D_I].$$

When c_1^2/D_I is large enough, we have $D_I/c_1^2 = o(1)$. Thus, one has

$$I_h(L) e^{-(1+o(1))\xi} \leq I_h\left(L - \frac{D_I}{c_1}\xi\right) \leq I_h(L) e^{-(1-o(1))\xi}, \quad \xi \in [0, c_1 L/D_I].$$

Integrating the above inequality over $(0, c_1 L/D_I)$, we obtain

$$I_h(L) \int_0^{\frac{c_1}{D_I}L} e^{-(1+o(1))\xi} \mathrm{d}\xi \leq \int_0^{\frac{c_1}{D_I}L} I_h\left(L - \frac{D_I}{c_1}\xi\right) \mathrm{d}\xi \leq I_h(L) \int_0^{\frac{c_1}{D_I}L} e^{-(1-o(1))\xi} \mathrm{d}\xi.$$

Since

$$\int_0^{\frac{c_1}{D_I}L} I_h\left(L - \frac{D_I}{c_1}\xi\right) \mathrm{d}\xi = \frac{c_1}{D_I} \int_0^L I_h(x) \mathrm{d}x,$$

one gets

$$\lim_{c_1/D_I \to \infty, c_1^2/D_I \to \infty} \frac{c_1}{D_I I_h(L)} \int_0^L I_h(x) \mathrm{d}x = 1.$$

Following the same logic, the other cases can also be verified. □

6. Discussion

In this paper, we considered a reaction-diffusion-advection vector-borne disease model with spatial heterogeneity, and investigated the effects of advection and diffusion terms on dynamics for the model through classifying the level set of basic reproduction ratio and the aggregation phenomenon of EE which to our knowledge may be the first attempt.

First, we established the well-posedness and threshold dynamics of (1.1). More precisely, the DFE E_0 is globally attractive if $\mathfrak{R}_0 \le 1$ and g.a.s. if $\mathfrak{R}_0 < 1$, and system (1.1) is persistent and possesses at least one EE if $\mathfrak{R}_0 > 1$ (see Lemma 2.4, Theorems 2.1 and 2.2). Note that since the no-flux boundary condition was adopted in the model, we need to transform it into homogeneous Neumann boundary condition via the transformation (2.1). Then the boundedness of solutions for (1.1) was proved thanks to the comparison principle. To compare with the existing results in [44], the monotonicity and asymptotic profiles of $\mathfrak{R}_0$ with and without advection terms were explored, respectively (see Propositions 3.1-3.2, Lemma 3.1 and Theorem 3.1). It should be pointed out that the study of the asymptotic properties of $\mathfrak{R}_0$ becomes more complicated under the advective environments than that in non-advective cases, but the results are also more impressive.

Next, we classified the level set of $\mathfrak{R}_0$ according to different situations, and discovered several interesting and important conclusions (see Theorems 4.1 and 4.2). Theorem 4.1 shown that when $\mathfrak{R}_{0a}^{\text{loc}} > 1$ and (H1) holds (i.e., the habitat and downstream end are located in a high-risk and low-risk area, respectively), there exist unique surfaces $\Upsilon_1(D_I, d_I)$ and $\Upsilon_2(D_I, d_I)$, such that E_0 is g.a.s. for $c_1 > \Upsilon_1(D_I, d_I)$ or $c_2 > \Upsilon_2(D_I, d_I)$, and system (1.1) admits at least one EE for $0 < c_1 < \Upsilon_1(D_I, d_I)$ or $0 < c_2 < \Upsilon_2(D_I, d_I)$ (see Fig. 2). Theorem 4.2 (I) indicated that when $\mathfrak{R}_{0a}^{\text{loc}} < 1$ and (H1) holds (i.e., the habitat and downstream end are located in low-risk sites), there are critical points $\widetilde{D}_I$ and $\widetilde{d}_I$ such that, for $(D_I, d_I) \in (0, \widetilde{D}_I) \times (0, \widetilde{d}_I)$, there exist unique surfaces $\Upsilon_3(D_I, d_I)$ and $\Upsilon_4(D_I, d_I)$, such that E_0 is g.a.s. for $c_1 > \Upsilon_3(D_I, d_I)$ or $c_2 > \Upsilon_3(D_I, d_I)$, and system (1.1) admits at least one EE for $0 < c_1 < \Upsilon_3(D_I, d_I)$ or $0 < c_2 < \Upsilon_4(D_I, d_I)$; For $(D_I, d_I) \in [\widetilde{D}_I, \infty) \times [\widetilde{d}_I, \infty)$, E_0 is g.a.s. for any $c_1 > 0$ and $c_2 > 0$ (see Fig. 3). Theorem 4.2 (II) implied that when $\mathfrak{R}_{0a}^{\text{loc}} < 1$ and (H2) holds (i.e., the habitat and downstream end are located in a low-risk and high-risk site, respectively), there are critical points $\widetilde{D}_I$ and $\widetilde{d}_I$ such that, for $(D_I, d_I) \in (0, \widetilde{D}_I] \times (0, \widetilde{d}_I]$, system (1.1) admits at least one EE for any $c_1 > 0$ and $c_2 > 0$; For $(D_I, d_I) \in (\widetilde{D}_I, \infty) \times (\widetilde{d}_I, \infty)$, there exist unique surfaces $\Upsilon_5(D_I, d_I)$ and $\Upsilon_6(D_I, d_I)$, such that system (1.1) admits at least one EE for $c_1 > \Upsilon_5(D_I, d_I)$ or $c_2 > \Upsilon_6(D_I, d_I)$, and E_0 is g.a.s. for $0 < c_1 < \Upsilon_5(D_I, d_I)$ or $0 < c_2 < \Upsilon_6(D_I, d_I)$ (see Fig. 4).

Finally, the aggregation behaviors of EE were studied (see Theorems 5.1 and 5.2). Theorems 5.1 and 5.2 implied that the densities of infected hosts or vectors will aggregate downstream end when the advection rates are large enough relative to their dispersal rates. It is technical to obtain the sub- and super-solutions for I_h and I_v with the help of the strong maximum principle of elliptic equations (see Lemmas 5.1 and 5.2). Nevertheless, we should point out that it is not conclusive to discuss the aggregation phenomenon of S_h, R_h and S_v because the transmission mechanism of vector-borne disease include two infection pathways and the external supplies are considered into model (1.1). We leave this issue as an open problem for future investigation.

Our findings complement the results of vector-borne disease in non-advective environments [4,23,41,44] and may provide several new clues for the investigation and control of the disease. Although the hypothesis in Lemma 4.1 and assumption $\hat{\beta}_1(x) \equiv \hat{\beta}_2(x)$ in this work are not very satisfactory, just for mathematical technique needs, it can be guaranteed the monotonicity of ϕ_2^1 and ϕ_4^1 in $(0, L)$ and uniqueness of c_1^* and c_2^* (see Propositions 4.1-4.2). As is known to all, many vector-borne diseases have incubation periods, and hosts and vectors can move randomly during the period [32,34]. This means that the infection thereby depends not only on the interaction at the current location and time, but also on the interaction of all possible locations at previous times, which usually can be described by a nonlocal incidence with a kernel function. Accordingly, it seems interesting and necessary to incorporate nonlocal effects and/or delay into vector-borne disease modeling. Future endeavors should explore the influences of nonlocality or delay in an advective heterogeneous environment.

Data availability

No data was used for the research described in the article.

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