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STABILITY ANALYSIS OF DELAY DIFFERENTIAL EQUATIONS WITH TWO DISCRETE DELAYS

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ABSTRACT. We use an algebraic method to derive a closed form for stability switching curves of delayed systems with two delays and delay independent coefficients for the first time. Furthermore, we provide some properties of these curves and stability switching directions. Our work is an extension of Gu *et al.*'s work [**7**] to a more general case using a different approach.

1 Introduction In the real world, delays appear in almost every procedure, and models with only one delay are often used under the assumption that other delays are small and insignificant to dynamical behaviors. However, this assumption may not be applicable in the case that a stability switch occurs even when an ignored delay is small. Therefore models with multiple delays are of great interest scientifically and mathematically. Realistic examples can be found in population interactions, neural networks, and SEIR epidemic models [**4, 6, 12, 14**]. A general theory on a special case of models with two delays was developed by Gu *et al.* [**7**], in which the characteristic function is of the following form:

$$D(\lambda) = P_0(\lambda) + P_1(\lambda)e^{-\tau_1\lambda} + P_2(\lambda)e^{-\tau_2\lambda}, \tag{1}$$

where τ_1 and τ_2 are the two delays, and

$$P_l(\lambda) = \sum_{k=0}^{n_l} p_{lk}\lambda^k, \quad l = 0, 1, 2$$

are polynomials with real coefficients.

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This method is useful to analyze functional differential equations (both neutral and retarded types) with only one population and delay independent parameters. However, in a more general circumstance, (1) is not applicable to delayed systems with multiple populations, which are more common as any species normally has connections with other species. In such a case, characteristic functions of the following form are frequently obtained for systems with delay independent coefficients [**11, 14**].

$$(2)\quad D(\lambda;\tau_1,\tau_2) = P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-\lambda\tau_2} + P_3(\lambda)e^{-\lambda(\tau_1+\tau_2)},$$

where τ_1 and τ_2 are the two delays in R_+, and

$$P_l(\lambda) = \sum_{k=0}^{n_l} p_{lk}\lambda^k, \quad l = 0,1,2,3$$

are polynomials with real coefficients.

The characteristic functions in [**5, 6, 11, 14**] are of the form (2) with special forms of P_i, $i = 0,1,2,3$. For example, the characteristic function in [**6**] has quadratic $P_0(\lambda)$, linear $P_1(\lambda), P_2(\lambda)$, and constant $P_3(\lambda)$. The food-chain model in [**5**] is a special case of (2) where $\tau_1 = \tau_2$ and $P_0(\lambda)$ is cubic. However, an analysis on (2) has not yet been fully explored, and this is the main focus of this paper.

Models with three or more delays have rarely been seen in mathematical biology. There exist some analytic efforts on systems with three discrete delays [**1**], but their applications seem quite limited. Usually they can only be applied to a scalar model with delay independent coefficients. In addition, as the number of delays increases, the dimension of stability switching surfaces increase, which makes stability much harder to determine. Instead of introducing multiple discrete delays, models with continuously distributed delays are also frequently encountered in mathematical biology.

In this paper, we first state some necessary assumptions on the characteristic function (2) to ensure it is a true characteristic function for a delay system. Next, we derive an explicit expression for the stability switching curves in the (τ_1,τ_2) plane in Section 3, and then we give a criterion to determine switching directions in Section 4. Finally, we apply our analytical results to the delayed Lotka-Volterra predator-prey model with two discrete delays.

2 Preliminary

The characteristic function we discuss throughout this paper is (2). One can see that the only difference from (1) is the appearance of the fourth term where the delays are mixed, and obviously if $P_3(\lambda) = 0$, (2) reduces to (1). Therefore, our analysis below is also applicable to (1).

To guarantee that (2) is a characteristic equation of some delayed system, we need some basic assumptions:

(i) Finite number of characteristic roots on $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ under the condition

$$\deg(P_0(\lambda)) \geq \max\{\deg(P_1(\lambda)), \deg(P_2(\lambda)), \deg(P_3(\lambda)\}. \tag{3}$$

(ii) Zero frequency: $\lambda = 0$ is not a characteristic root for any τ_1 and τ_2, i.e.,

$$P_0(0) + P_1(0) + P_2(0) + P_3(0) \neq 0.$$

(iii) The polynomials $P_0,\ P_1,\ P_2$ and P_3 have no common zeros, i.e., $P_0,\ P_1,\ P_2$ and P_3 are coprime polynomials.

(iv) P_k's satisfy

$$\lim_{\lambda\to\infty}\left(\left|\frac{P_1(\lambda)}{P_0(\lambda)}\right| + \left|\frac{P_2(\lambda)}{P_0(\lambda)}\right| + \left|\frac{P_3(\lambda)}{P_0(\lambda)}\right|\right) < 1.$$

If (i) is violated, then the characteristic equation can never be stable since there are infinitely many roots with positive real parts [**2**].

If (ii) is violated, then $D(0, \tau_1, \tau_2) \equiv 0$ for all $(\tau_1, \tau_2) \in \mathbb{R}_+^2$, and therefore the characteristic function is always unstable.

Assumption (iii) is to ensure the considered characteristic equation has the lowest degree and is irreducible.

Assumption (iv) is to exclude large oscillations, i.e., feasible ω's for (4) are bounded (see Section 3). In reality, if feasible ω can be arbitrarily large, $D(\lambda)$ will be highly sensitive to the delays τ_1 and τ_2. For retarded type delay equations, this assumption is automatically satisfied.

3 Stability switching curves

Lemma 3.1. *As (τ_1, τ_2) varies continuously in $\mathbb{R}_+^2$, the number of characteristic roots (with multiplicity counted) of $D(\lambda; \tau_1, \tau_2)$ on $\mathbb{C}_+$ can change only if a characteristic root appears on or cross the imaginary axis.*

Proof of this lemma can be found in any book on functional differential equations, for example, [**8, 13**].

From this lemma, to study stability switching, we seek purely imaginary characteristic roots. Since $\lambda \neq 0$ by assumption (ii), and roots of a real function always come in conjugate pairs, we assume $\lambda = i\omega$ ($\omega > 0$). Substituting this into (2), we get

$$D(i\omega;\tau_1,\tau_2) = \left(P_0(i\omega) + P_1(i\omega)e^{-i\omega\tau_1}\right) + \left(P_2(i\omega) + P_3(i\omega)e^{-i\omega\tau_1}\right)e^{-i\omega\tau_2}. \tag{4}$$

Since $|e^{-i\omega\tau_2}| = 1$, we have

$$|P_0 + P_1 e^{-i\omega\tau_1}| = |P_2 + P_3 e^{-i\omega\tau_1}|, \tag{5}$$

which is equivalent to

$$(P_0 + P_1 e^{-i\omega\tau_1})(\overline{P}_0 + \overline{P}_1 e^{i\omega\tau_1}) = (P_2 + P_3 e^{-i\omega\tau_1})(\overline{P}_2 + \overline{P}_3 e^{i\omega\tau_1}).$$

After simplification, we have

$$|P_0|^2 + |P_1|^2 + 2\mathrm{Re}\,(P_0\overline{P}_1)\cos(\omega\tau_1) - 2\mathrm{Im}\,(P_0\overline{P}_1)\sin(\omega\tau_1) = |P_2|^2 + |P_3|^2 + 2\mathrm{Re}\,(P_2\overline{P}_3)\cos(\omega\tau_1) - 2\mathrm{Im}\,(P_2\overline{P}_3)\sin(\omega\tau_1).$$

Thus,

$$|P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2 = 2A_1(\omega)\cos(\omega\tau_1) - 2B_1(\omega)\sin(\omega\tau_1), \tag{6}$$

where

$$A_1(\omega) = \mathrm{Re}\,(P_2\overline{P}_3) - \mathrm{Re}\,(P_0\overline{P}_1),$$
$$B_1(\omega) = \mathrm{Im}\,(P_2\overline{P}_3) - \mathrm{Im}\,(P_0\overline{P}_1).$$

If there is some ω such that $A_1(\omega)^2 + B_1(\omega)^2 = 0$, then

$$A_1(\omega) = B_1(\omega) = 0 \Longleftrightarrow P_0\overline{P}_1 = P_2\overline{P}_3. \tag{7}$$

The right hand side of (6) is 0 with any τ_1, and

$$|P_0|^2 + |P_1|^2 = |P_2|^2 + |P_3|^2. \tag{8}$$

Therefore, if there is an ω such that both (7) and (8) are satisfied, then all $\tau_1 \in \mathbb{R}_+$ are solutions of (5).

If $A_1(\omega)^2 + B_1(\omega)^2 > 0$, then there exists some continuous function $\phi_1(\omega)$ such that

$$A_1(\omega) = \sqrt{A_1(\omega)^2 + B_1(\omega)^2}\cos(\phi_1(\omega)),$$
$$B_1(\omega) = \sqrt{A_1(\omega)^2 + B_1(\omega)^2}\sin(\phi_1(\omega)).$$

Indeed,

$$\phi_1(\omega) = \arg\{P_2\bar{P}_3 - P_0\bar{P}_1\}.$$

Therefore, (6) becomes

$$(9)\quad |P_0|^2+|P_1|^2-|P_2|^2-|P_3|^2 = 2\sqrt{A_1(\omega)^2 + B_1(\omega)^2}\cos(\phi_1(\omega)+\omega\tau_1).$$

Obviously, a sufficient and necessary condition for the existence of $\tau_1 \in \mathbb{R}_+$ satisfying the above equation is

$$(10)\qquad \big||P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2\big| \leq 2\sqrt{A_1^2 + B_1^2}.$$

Denote Ω^1 to be $\omega \in \mathbb{R}_+$ satisfying (10). One should notice that (10) also includes the case $A_1^2 + B_1^2 = 0$ which leads to (7) and (8).

Let

$$\cos(\psi_1(\omega)) = \frac{|P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2}{2\sqrt{A_1^2 + B_1^2}}, \quad \psi_1 \in [0, \pi].$$

We have

$$(11)\qquad \tau_{1,n_1}^{\pm}(\omega) = \frac{\pm\psi_1(\omega) - \phi_1(\omega) + 2n_1\pi}{\omega}, \quad n_1 \in \mathbb{Z}.$$

All the formulas in these steps can be obtained explicitly. Once we get $\tau_1(\omega)$ given by (11), substitute it into (4) and we get an explicit formula for $\tau_2(\omega)$ unconditionally with each $\omega \in \Omega^1$, i.e.,

$$(12)\qquad \tau_{2,n_2}^{\pm}(\omega) = \frac{1}{\omega}\arg\left\{-\frac{P_2 + P_3 e^{-i\omega\tau_1^{\pm}}}{P_0 + P_1 e^{-i\omega\tau_1^{\pm}}}\right\} + 2n_2\pi, \quad n_2 \in \mathbb{Z}.$$

Thus the stability crossing curves are

$$\mathcal{T} := \{(\tau^{\pm}_{1,n_1}(\omega), \tau^{\pm}_{2,n_2}(\omega)) \in \mathbb{R}^2_+ : \omega \in \Omega^1,\ n_1, n_2 \in \mathbb{Z}\}. \tag{13}$$

Another way to find τ_2 is to analyze τ_2 similarly to the analysis of τ_1, which gives

$$\tau^{\pm}_{2,n_2} = \frac{\pm\psi_2(\omega) - \phi_2(\omega) + 2n_2\pi}{\omega}, \quad n_2 \in \mathbb{Z}, \tag{14}$$

where

$$\begin{aligned}
\cos(\psi_2(\omega)) &= \frac{|P_0|^2 - |P_1|^2 + |P_2|^2 - |P_3|^2}{2\sqrt{A_2^2 + B_2^2}}, \quad \psi_2 \in [0, \pi],\\
A_2(\omega) &= \sqrt{A_2(\omega)^2 + B_2(\omega)^2}\cos(\phi_2(\omega)),\\
B_2(\omega) &= \sqrt{A_2(\omega)^2 + B_2(\omega)^2}\sin(\phi_2(\omega)),\\
A_2(\omega) &= \mathrm{Re}\,(P_1\overline{P}_3) - \mathrm{Re}\,(P_0\overline{P}_2),\\
B_2(\omega) &= \mathrm{Im}\,(P_1\overline{P}_3) - \mathrm{Im}\,(P_0\overline{P}_2),
\end{aligned}$$

with the condition on ω:

$$\big||P_0|^2 - |P_1|^2 + |P_2|^2 - |P_3|^2\big| \le 2\sqrt{A_2^2 + B_2^2}, \tag{15}$$

which defines a region Ω^2.

By squaring both sides of the two conditions (10) and (15), one can show that (10) and (15) are equivalent. Thus,

$$\Omega := \Omega^1 = \Omega^2.$$

We call Ω the crossing set.

Lemma 3.2. Ω *consists of a finite number of intervals of finite length.*

Proof. Let

$$F(\omega) := \big(|P_0|^2 + |P_1|^2 - |P_2|^2 - |P_3|^2\big)^2 - 4(A_1^2 + B_1^2), \quad \omega \ge 0.$$

By assumption (ii), we have $F(0) \ne 0$. In addition, by assumptions (i) and (iv), $F(+\infty) = +\infty$. Thus $F(\omega)$ has a finite number of roots on $\mathbb{R}_+$.

If $F(0) > 0$, then F has roots $0 < a_1 < b_1 \le a_2 < b_2 < \cdots \le a_N < b_N < +\infty$, and

$$\Omega = \bigcup_{k=1}^{N} \Omega_k, \quad \Omega_k = [a_k, b_k].$$

If $F(0) < 0$, then F has roots $0 < b_1 \le a_2 < b_2 \le a_3 < b_3 < \cdots \le a_N < b_N < +\infty$, and

$$\Omega = \bigcup_{k=1}^{N} \Omega_k, \quad \Omega_1 = (0, b_1], \ \Omega_k = [a_k, b_k] \ (k \ge 2).$$

We allow $b_{k-1} = a_k$ with the concern that a_k may be a root with even multiplicity. □

For any Ω_k, we have a restriction on the range of $\phi_i(\omega), \ i = 1, 2$. We require $\phi_i(\omega)$ to be the smallest continuous branch with the property that there exists an $\omega_i \in \Omega_k$, such that

$$\phi_i(\omega_i) > 0.$$

Therefore, n_i has a lower bound, denoted by $L_{i,k}$.

However, by (12) one should obtain either τ^+_{2,n_2} or τ^-_{2,n_2} (but not both) for a given τ^+_{1,n_1}, and similarly for τ^+_{1,n_1}. By tedious computation (or by MATHEMATICA/MAPLE), one can verify that when $\tau_1 = \tau^+_{1,n_1}(\omega)$, we have $\tau_2 = \tau^-_{2,n_2}(\omega)$, and when $\tau_1 = \tau^-_{1,n_1}(\omega)$, we have $\tau_2 = \tau^+_{2,n_2}(\omega)$. Therefore,

$$\mathcal{T} = \bigcup_{\substack{k=1,2,\ldots,N \\ n_1 \ge L_{1,k}, L_{1,k}+1, \cdots \\ n_2 \ge L_{2,k}, L_{2,k}+1, \cdots}} \mathcal{T}^{\pm k}_{n_1,n_2} \bigcap \mathbb{R}^2_+, \tag{16}$$

$$\mathcal{T}^{\pm k}_{n_1,n_2} = \left\{ \left(\frac{\pm\psi_1(\omega) - \phi_1(\omega) + 2n_1\pi}{\omega}, \right.\right. \left.\left. \frac{\mp\psi_2(\omega) - \phi_2(\omega) + 2n_2\pi}{\omega} \right) : \omega \in \Omega_k \right\}. \tag{17}$$

Obviously, $\mathcal{T}^{\pm k}_{n_1,n_2}$ is continuous in $\mathbb{R}^2$.

Since $f(a_k) = f(b_k) = 0$, we have

$$\cos(\psi_i(a_k)) = \delta_i^a \pi, \quad \cos(\psi_i(b_k)) = \delta_i^b \pi,$$

where $\delta_i^a, \delta_i^b = 0, 1$, $i = 1, 2$. One can easily verify that

$$\begin{aligned}(18)\qquad &\big(\tau_{1,n_1}^{+k}(a_k), \tau_{2,n_1}^{-k}(a_k)\big) = \big(\tau_{1,n_1+\delta_1^a}^{-k}(a_k), \tau_{2,n_2-\delta_2^a}^{+k}(a_k)\big),\\ &\big(\tau_{1,n_1}^{+k}(b_k), \tau_{2,n_1}^{-k}(b_k)\big) = \big(\tau_{1,n_1+\delta_1^b}^{-k}(b_k), \tau_{2,n_2-\delta_2^b}^{+k}(b_k)\big).\end{aligned}$$

Therefore, $\mathcal{T}_{n_1,n_2}^{+k}$ connects $\mathcal{T}_{n_1+\delta_1^a,n_2-\delta_2^a}^{-k}$ and $\mathcal{T}_{n_1+\delta_1^a,n_2-\delta_2^b}^{-k}$ at its two ends, and we have the following theorem.

Theorem 3.1. *$\mathcal{T}$ defined in* (16) *is the set of all stability switching curves on the (τ_1, τ_2)-plane for* (2) *. Furthermore, if $(\delta_1^a, \delta_2^a) = (\delta_1^b, \delta_2^b)$, then $\mathcal{T}_{n_1,n_2}^{+k}$ and $\mathcal{T}_{n_1+\delta_1^a,n_2-\delta_2^a}^{-k}$ form a loop on $\mathbb{R}^2$, and $\mathcal{T}$ is a set of closed continuous curves (Class I); while if $(\delta_1^a, \delta_2^a) \neq (\delta_1^b, \delta_2^b)$, $\mathcal{T}$ is a set of continuous curves with their two end points either on the axises or extending to infinity on the $\mathbb{R}_+^2$ region (Class II).*

4 Stability: crossing directions Let $\lambda = \sigma + i\omega$. Then by the implicit function theorem, τ_1, τ_2 can be expressed as functions of σ and ω under some non-singular condition. For symbolic convenience, denote $\tau_3 := \tau_1 + \tau_2$.

$$\begin{aligned}(19)\qquad R_0 &:= \left.\frac{\partial \mathrm{Re}\, D(\lambda; \tau_1, \tau_2)}{\partial \sigma}\right|_{\lambda=i\omega}\\ &= \mathrm{Re}\,\{P_0'(i\omega) + \sum_{k=1}^{3} \left(P_k'(i\omega) - \tau_k P_k(i\omega)\right) e^{-i\omega\tau_k}\},\end{aligned}$$

$$\begin{aligned}(20)\qquad I_0 &:= \left.\frac{\partial \mathrm{Im}\, D(\lambda; \tau_1, \tau_2)}{\partial \sigma}\right|_{\lambda=i\omega}\\ &= \mathrm{Im}\,\{P_0'(i\omega) + \sum_{k=1}^{3} \left(P_k'(i\omega) - \tau_k P_k(i\omega)\right) e^{-i\omega\tau_k}\}.\end{aligned}$$

Similarly, one can verify that

$$(21)\qquad \left.\frac{\partial \mathrm{Re}\, D(\lambda; \tau_1, \tau_2)}{\partial \omega}\right|_{\lambda=i\omega} = -I_0,$$

$$(22)\qquad \left.\frac{\partial \mathrm{Im}\, D(\lambda; \tau_1, \tau_2)}{\partial \omega}\right|_{\lambda=i\omega} = R_0.$$

We also have

$$R_l := \left.\frac{\partial \mathrm{Re}\, D(\lambda;\tau_1,\tau_2)}{\partial \tau_l}\right|_{\lambda=i\omega} = \mathrm{Re}\,\{-i\omega\left(P_l(i\omega)e^{-i\omega\tau_l} + P_3(i\omega)e^{-i\omega(\tau_1+\tau_2)}\right)\}, \tag{23}$$

$$I_l := \left.\frac{\partial \mathrm{Im}\, D(\lambda;\tau_1,\tau_2)}{\partial \tau_l}\right|_{\lambda=i\omega} = \mathrm{Im}\,\{-i\omega\left(P_l(i\omega)e^{-i\omega\tau_l} + P_3(i\omega)e^{-i\omega(\tau_1+\tau_2)}\right)\}, \tag{24}$$

where $l = 1, 2$.

From the derivation, $\mathcal{T}^{\pm k}_{n_1,n_2}$ are piecewise differentiable. By the implicit function theory, we have

$$\Delta(\omega) := \left.\begin{pmatrix} \dfrac{\partial\tau_1}{\partial\sigma} & \dfrac{\partial\tau_1}{\partial\omega} \\ \dfrac{\partial\tau_2}{\partial\sigma} & \dfrac{\partial\tau_2}{\partial\omega} \end{pmatrix}\right|_{\sigma=0,\omega\in\Omega} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 & -I_0 \\ I_0 & R_0 \end{pmatrix}. \tag{25}$$

The implicit function theorem applies as long as

$$\det \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix} = R_1 I_2 - R_2 I_1 \neq 0.$$

For any crossing curve $\mathcal{T}^{\pm k}_{n_1,n_2}$, we call the direction of the curve corresponding to increasing $\omega \in \Omega_k$ the *positive direction*, and the region on the left-hand (right-hand) side when we move in the positive direction of the curve *the region on the left* (*right*). Since the tangent vector of $\mathcal{T}^{\pm k}_{n_1,n_2}$ along the positive direction is $(\partial\tau_1/\partial\omega, \partial\tau_2/\partial\omega)$, the normal vector of $\mathcal{T}^{\pm k}_{n_1,n_2}$ pointing to the right region is $(\partial\tau_2/\partial\omega, -\partial\tau_1/\partial\omega)$. As we know, a pair of complex characteristic roots across the imaginary axis to the right on the complex plane as σ increases from negative to positive through 0, thus (τ_1, τ_2) moves along the direction $(\partial\tau_1/\partial\sigma, \partial\tau_2/\partial\sigma)$. As a consequence, we can conclude that if

$$\delta(\omega) := \left(\frac{\partial\tau_1}{\partial\sigma}, \frac{\partial\tau_2}{\partial\sigma}\right) \cdot \left(\frac{\partial\tau_2}{\partial\omega}, -\frac{\partial\tau_1}{\partial\omega}\right) = \frac{\partial\tau_1}{\partial\sigma}\frac{\partial\tau_2}{\partial\omega} - \frac{\partial\tau_2}{\partial\sigma}\frac{\partial\tau_1}{\partial\omega} = \det\Delta(\omega) > 0, \tag{26}$$

the region on the right of $\mathcal{T}^{\pm k}_{n_1,n_2}$ has two more characteristic roots with positive real parts. On the other hand, if inequality (26) is reversed, then the region on the left has two more characteristic roots with positive real parts.

Since

$$\det\begin{pmatrix} R_0 & -I_0 \\ I_0 & R_0 \end{pmatrix} = R_0^2 + I_0^2 \geq 0,$$

we have

$$\operatorname{sign}\delta(\omega) = \operatorname{sign}\{R_1 I_2 - R_2 I_1\}, \tag{27}$$

if either $R_0 \neq 0$ or $I_0 \neq 0$. For any $(\tau_1^{\pm}(\omega), \tau_2^{\mp}(\omega)) \in \mathcal{T}^{\pm k}_{n_1,n_2}$, we have

$$P_2(i\omega)e^{i\omega(\tau_1^{\pm}-\tau_2^{\mp})} = -P_0(i\omega)e^{i\omega\tau_1^{\pm}} - P_1(i\omega) - P_3(i\omega)e^{-i\omega\tau_2^{\mp}}.$$

One can verify that

$$\begin{aligned} R_1 I_2 - R_2 I_1 \quad & \\ = \operatorname{Im}\Big\{ & \overline{-i\omega\big(P_1 e^{-i\omega\tau_1^{\pm}} + P_3 e^{-i\omega(\tau_1^{\pm}+\tau_2^{\mp})}\big)}(-i\omega) \\ & \times \big(P_2 e^{-i\omega\tau_2^{\mp}} + P_3 e^{-i\omega(\tau_1^{\pm}+\tau_2^{\mp})}\big)\Big\} \\ = \omega^2 \operatorname{Im}\Big\{ & \overline{P}_1 P_2 e^{i\omega(\tau_1^{\pm}-\tau_2^{\mp})} + \overline{P}_1 P_3 e^{-i\omega\tau_2^{\mp}} + P_2 \overline{P}_3 e^{i\omega\tau_1^{\pm}}\Big\} \\ = \omega^2 \operatorname{Im}\Big\{ & \overline{P}_1(-P_0 e^{i\omega\tau_1^{\pm}} - P_1 - P_3 e^{-i\omega\tau_2^{\mp}}) \\ & + \overline{P}_1 P_3 e^{-i\omega\tau_2^{\mp}} + P_2 \overline{P}_3 e^{i\omega\tau_1^{\pm}}\Big\} \\ = \omega^2 \operatorname{Im}\{ & (P_2\overline{P}_3 - P_0\overline{P}_1) e^{i\omega\tau_1^{\pm}}\} \\ = \omega^2 \operatorname{Im}\{ & |P_2\overline{P}_3 - P_0\overline{P}_1| e^{\phi_1} e^{i\omega\tau_1^{\pm}}\} \\ = \pm\omega^2 & |P_2\overline{P}_3 - P_0\overline{P}_1| \sin\psi_1. \end{aligned} \tag{28}$$

Hence,

$$\delta(\omega \in \mathring{\Omega}_k) = \pm\operatorname{sign}(\omega^2 |P_2\overline{P}_3 - P_0\overline{P}_1| \sin\psi_1) = \pm 1, \tag{29}$$

since $\psi_1(\mathring{\Omega}_k) \subset (0,\pi)$. Here, $\mathring{\Omega}_k$ denotes the interior of Ω_k.

As we know from (18), two connected curves, $\mathcal{T}^{+k}_{n_1,n_2}$ and $\mathcal{T}^{-k}_{n_1+\delta_1,n_2-\delta_2}$, are orientated in opposite directions. On the other hand, (29) shows that they have different stability switching directions. As a result, as we move along these continuous curves, stability switching directions are consistent, i.e., there always exist two more characteristic roots with positive real parts on the right or on the left.

Theorem 4.1. *For any $k = 1, 2, \cdots, N$, we have*

$$\delta(\omega \in \mathring{\Omega}_k) \equiv \pm 1, \quad \forall\, (\tau_1(\omega), \tau_2(\omega)) \in \mathcal{T}^{\pm k}_{n_1,n_2}.$$

Therefore, the region on the left of $\mathcal{T}^{+k}_{n_1,n_2}$ ($\mathcal{T}^{-k}_{n_1,n_2}$) has two more (less) characteristic roots with positive real parts.

If we know the number of characteristic roots with positive real parts when $\tau_1 = \tau_2 = 0$, we can use criterion (26) to find the number of characteristic roots with positive real parts for any (τ_1, τ_2) in $\mathbb{R}^2_+$. In this way, stability based on the characteristic equation is completely known.

5 Example: Lotka-Volterra predator-prey model with two delays Consider the following Lotka-Volterra model:

$$\begin{aligned} x'(t) &= rx(t)\left(1 - \frac{x(t-\tau_1)}{K}\right) - f(x(t))y(t), \\ y'(t) &= \gamma e^{-d_j\tau_2} f(x(t-\tau_2))y(t-\tau_2) - dy(t). \end{aligned} \tag{30}$$

In order to apply our method, $d_j = 0$ is required. Indeed, in some cases, the death rate of the juvenile (d_j) could be extremely small, for instance, human beings in developed countries. To make the analysis easy, we choose the simple Holling type I response function $f(x) = bx$ to illustrate our method. Indeed, a similar analysis can be done with Holling type II/III response function, or any other type. For simplicity, we nondimensionalize the system to obtain

$$\begin{aligned} x'(t) &= x(t)\left(1 - x(t-\tau_1)\right) - x(t)y(t), \\ y'(t) &= cx(t-\tau_2)y(t-\tau_2) - dy(t), \end{aligned} \qquad d < c < 1. \tag{31}$$

Our interest here is on the stability of the interior equilibrium $P = (d/c, (c-d)/c)$. Linearization around P gives

$$\begin{aligned} x'(t) &= -\frac{d}{c}\,x(t-\tau_1) - \frac{d}{c}\,y(t), \\ y'(t) &= (c-d)x(t-\tau_2) - dy(t) + dy(t-\tau_2). \end{aligned} \tag{32}$$

By substituting a solution of the type $(x(t), y(t)) = (c_1, c_2)e^{\lambda t}$, we obtain

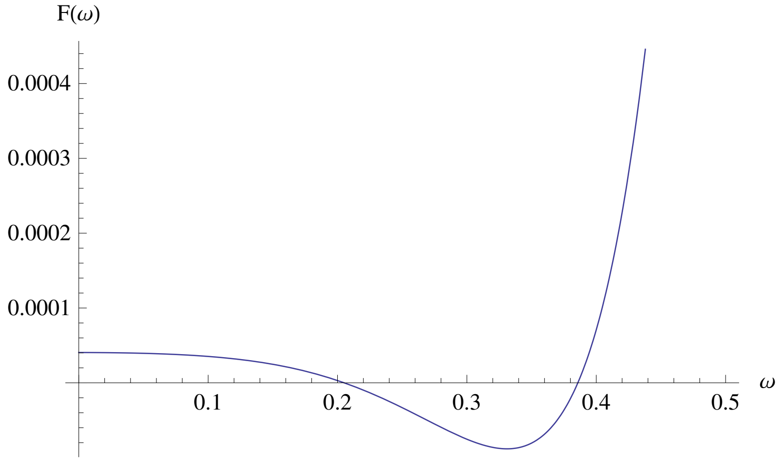

FIGURE 1: Graph of $F(\omega)$.

the characteristic function

$$D(\lambda;\tau_1,\tau_2)=\lambda^2+d\lambda+\left(\frac{d}{c}\lambda+\frac{d^2}{c}\right)e^{-\lambda\tau_1} - \left(d\lambda+\frac{d^2}{c}-d\right)e^{-\lambda\tau_2}-\frac{d^2}{c}e^{-\lambda(\tau_1+\tau_2)}. \tag{33}$$

For simulations, we choose $c=0.6,\ d=0.1$. $F(\omega)$ has only two roots $a_1=0.2052,\ b_1=0.3858$ (see Figure 1), and

$$\delta_1^a=-1,\quad \delta_2^a=1,\quad \delta_2^b=1,\quad \delta_2^b=1.$$

By Theorem 3.1, the switching curves are of class I, which is shown in Figure 2. With the aid of Figure 2, to find the crossing directions of the characteristic roots as τ_1 and τ_2 vary, one only needs to analyze on the first intercept on the τ_2 axes. Crossing directions for the other curves can be easily deduced by Theorem 4.1. A simple calculation can verify that when $\tau_1=\tau_2=0$, the interior equilibrium is stable, i.e., no characteristic roots have positive real parts. Hence from Figure (2), the interior equilibrium is stable if and only if (τ_1,τ_2) is on the small bottom-left region of the (τ_1,τ_2)-plane. As τ_1 and τ_2 increase, we see a trend that there are more and more charactersitic roots with positive

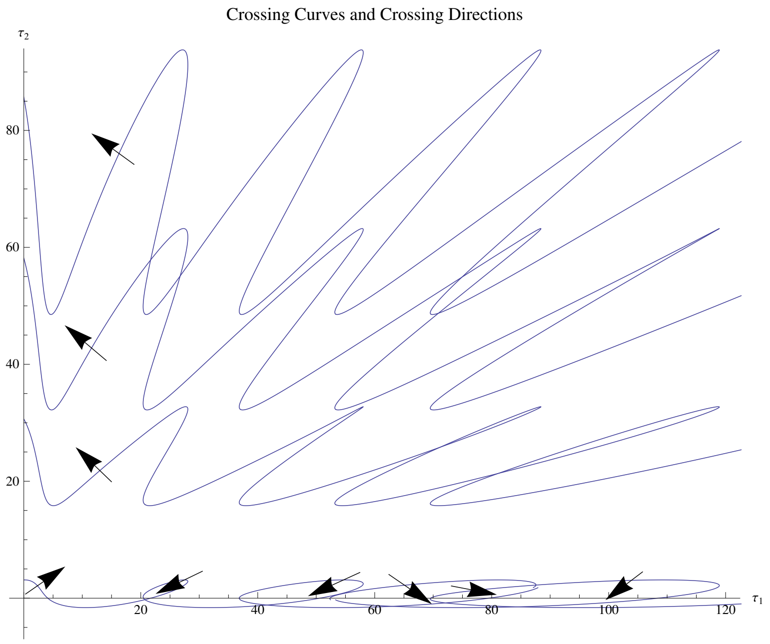

FIGURE 2: Plot of Switching curves. Arrows are used to illustrate the crossing directions, that is, the region on the end of an arrow has two more characteristic roots with positive real parts. Curves on bottom-right quadrant are shown so as to reveal how a continuous curve behaves. However in a delayed system, only curves in the first quadrant are of interest.

real parts. Furthermore, on the τ_1-axis, i.e., when $\tau_2 = 0$, characteristic roots cross the imaginary axis to the right and left alternatively in the complex plane, with appearance and disappearance of periodic solutions, which results in Hopf bifurcations and may lead to multiple stable limit cycles [**9, 10**].

6 Concluding remarks If $P_3 \equiv 0$, we expect to reduce our results to those obtained in Gu et al. [**7**]. Indeed, $A_1 + iB_1 = P_0\overline{P}_1$, and this together with (10) gives the range of feasible ω:

$$(|P_0| - |P_1|)^2 \leq |P_2|^2, \qquad |P_2|^2 \leq (|P_0|^2 + |P_1|)^2,$$

which basically says that P_0, P_1, P_2 form a closed triangle region or a degenerated triangle region. This is exactly the same as the condition given by Gu et al.. In addition, one can verify that the formula of crossing curves given in (16) is the same as formula (3.5) and (3.6) in [**7**], though the methods used there are different. Note that the results in this paper cannot be obtained from the geometric method introduced in [**7**].

It is hard to extend our method to the case when P_i's are delay dependent. Research on one delay with delay dependent coefficients has been done in Beretta and Kuang [**3**], which gives an efficient algorithm to determine the stability. However, in the two-delay case, even for the following simple characteristic equation

$$P_0(\lambda;\tau_1) + P_1(\lambda;\tau_1)e^{-\lambda\tau_1} + P_2(\lambda;\tau_1)e^{-\lambda\tau_2} = 0,$$

there is no efficient method so far. The difficulty here is the ω, τ_1 mix in a transcendental manner after removing τ_2, i.e.

$$|P_0(i\omega;\tau_1) + P_1(i\omega;\tau_1)e^{-i\omega\tau_1}| = |P_2(i\omega;\tau_1)|,$$

which makes it complicated to compute one given the other, as multiple or even infinity many choices can occur. Furthermore, the range of ω cannot be simply determined, and it may depend on τ_1. A novel method is required for this case.

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