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Traveling waves for a diffusive mosquito-borne epidemic model with general incidence

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Abstract. In this paper, we obtain the complete information about the existence and nonexistence of traveling wave solution (TWS) for a reaction–diffusion model of mosquito-borne disease with general incidence and constant recruitment. We find that the basic reproduction ratio $\mathfrak{R}_0$ of the corresponding kinetic system and the minimal wave speed c_* are thresholds to determine the existence of TWS. With the aid of limiting arguments and Lyapunov approach, it is demonstrated that the system possesses a nontrivial TWS with wave speed $c \geq c_*$ connecting the disease-free equilibrium and endemic equilibrium when $\mathfrak{R}_0 > 1$. When $\mathfrak{R}_0 \leq 1$ and $c > 0$, the nonexistence of nontrivial TWS is obtained by contradiction. By means of a rather ingenious method that is easier to understand than Laplace transform, we show that there is no nontrivial TWS when $\mathfrak{R}_0 > 1$ and $0 < c < c_*$. Numerically, we perform simulations to verify the analytical results and explore the sensitivity of the speed c_* on parameters. The sensitivity results show that the parameters related to mosquitoes have a greater impact on c_*.

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Keywords. Reaction–diffusion model, Mosquito-borne disease, Traveling wave solution, Basic reproduction ratio, Minimal wave speed.

1. Introduction

Mosquito-borne disease, a disease that the pathogens are transmitted to humans through mosquitoes, has become one of the most serious challenges threatening human health [1]. Some common such diseases include malaria, dengue fever, Zika and chikungunya. Due to the severity of diseases, it is necessary to study the spread of mosquito-borne diseases with various mathematical models (such as ordinary differential equations (ODE) [7,38], delay differential equations (DDE) [18,30,40], reaction–diffusion (R-D) equations [24,25,29,32,37] and so on). In epidemiology, spatial effects have been extensively introduced into models to delve into the geographical spread of infectious diseases. Generally, an epidemic model with spatial effects will generate an epidemic wave, which connects the equilibria of the model, and such epidemic wave is described by TWS propagating at a certain speed [4,39]. It seems thus meaningful to analyze TWS so as to better understand the spatial spread of mosquito-borne diseases [11,21].

The importance of TWS in infectious diseases prompted many researchers devote themselves to the research of it, and so plenty of excellent works have been done in the past decades, see for [5,13,14,16,33, 35,36,39,41,42] and references therein. However, as far as we know, few studies seem to focus on the TWS of mosquito-borne disease models (e.g. [4,8,15,17,26]). In fact, due to the complexity of model caused by the transmission mechanism of diseases (the virus is not transmitted directly from human to human, but through the bite of infected mosquitoes), the study of TWS for mosquito-borne disease models has been quite limited till now. In 2006, Lewis et al. [15] proposed a R-D West Nile virus (WNv) (a mosquito-borne disease) model with standard incidence and studied the existence of TWS of the simplified version for the model. In 2017, Lin and Zhu [17] established a R-D model with free boundary and standard incidence for WNv, and discussed the existence of TWS of the corresponding simplified spatial model.

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More recently, Wang et al. [26] investigated the TWS of a R-D vector-borne disease model with nonlocal effects and distributed delay. Denu et al. [4] proposed a deterministic vector-host epidemic model with bilinear incidence and constant recruitment. They established the existence and nonexistence of TWS for the model. In [4], the authors assumed that susceptible vectors (hosts) have the same diffusion rates as infected vectors (hosts). It should be pointed out that the capacity of activity for susceptible individuals is usually stronger than that of infected individuals according to [39].

Note that a fair amount of mosquito-borne disease models mainly adopted bilinear incidence [4,32], standard incidence [7] or saturation incidence [20]. However, inspired by the ideas in [3,6,42], the nonlinear (general) incidence is better to give a reasonable qualitative description for the disease dynamics. On the other hand, to investigate the dynamics more comprehensively, it seems necessary to incorporate the external supplies (recruitment) of individuals and mosquitoes into the modeling of mosquito-borne diseases [29,32].

Motivated by above analysis, in this paper, we study the following mosquito-borne epidemic model with general incidence rates and constant recruitment

$$\begin{cases} \partial_t S_h(t,x) = D_S \Delta S_h(t,x) + \Lambda - f_1(S_h, I_v)(t,x) - \mu_1 S_h(t,x), \\ \partial_t I_h(t,x) = D_I \Delta I_h(t,x) + f_1(S_h, I_v)(t,x) - (\mu_1 + d_1 + \alpha_1) I_h(t,x), \\ \partial_t R_h(t,x) = D_R \Delta R_h(t,x) + \alpha_1 I_h(t,x), \\ \partial_t S_v(t,x) = d_S \Delta S_v(t,x) + M - f_2(S_v, I_h)(t,x) - \mu_2 S_v(t,x), \\ \partial_t I_v(t,x) = d_I \Delta I_v(t,x) + f_2(S_v, I_h)(t,x) - (\mu_2 + d_2) I_v(t,x), \end{cases} \tag{1.1}$$

wherein $S_h := S_h(t,x)$, $I_h := I_h(t,x)$ and $R_h := R_h(t,x)$ are the spatial densities of susceptible, infectious and recovered individuals, and $S_v := S_v(t,x)$ and $I_v := I_v(t,x)$ are the spatial densities of susceptible and infectious mosquitoes at time t and location x, respectively. The diffusion rates of S_h, I_h, R_h and S_v, I_v are denoted by D_S, D_I, D_R and d_S, d_I, respectively. The recruitment and natural death rate of individuals and mosquitoes are represented by Λ, μ_1 and M, μ_2 respectively. The d_1 and d_2 represent the disease-induced death rates of individuals and mosquitoes, respectively. The recovery rate of infectious individuals is denoted by α_1. The $f_1(S_h, I_v)$ and $f_2(S_v, I_h)$ mean the disease transmission functions. For the sake of simplicity, let $\gamma_1 := \mu_1 + d_1 + \alpha_1$ and $\gamma_2 := \mu_2 + d_2$. By the decoupling, it is sufficient to discuss the following system

$$\begin{cases} \partial_t S_h = D_S \Delta S_h + \Lambda - f_1(S_h, I_v) - \mu_1 S_h, \\ \partial_t I_h = D_I \Delta I_h + f_1(S_h, I_v) - \gamma_1 I_h, \\ \partial_t S_v = d_S \Delta S_v + M - f_2(S_v, I_h) - \mu_2 S_v, \\ \partial_t I_v = d_I \Delta I_v + f_2(S_v, I_h) - \gamma_2 I_v. \end{cases} \tag{1.2}$$

Throughout this paper, unless otherwise indicated, we make the following assumptions:

(P1) $f_1(S_h, I_v)$, $f_2(S_v, I_h) \in C^2(\mathbb{R}_+ \times \mathbb{R}_+)$, and the partial derivatives $\partial_{S_h} f_1(S_h, I_v)$, $\partial_{I_v} f_1(S_h, I_v)$, $\partial_{S_v} f_2(S_v, I_h)$ and $\partial_{I_h} f_2(S_v, I_h)$ are positive for all $S_h, I_h, S_v, I_v > 0$.

(P2) $f_1(S_h, I_v) = 0$ if and only if (iff) $S_h I_v = 0$, and $f_2(S_v, I_h) = 0$ iff $S_v I_h = 0$; $\partial^2_{I_v} f_1(S_h, I_v) \le 0$ and $\partial^2_{I_h} f_2(S_v, I_h) \le 0$.

(P3) All coefficients of model (1.1) are positive, and $D_S \ge D_I$, $d_S \ge d_I$.

Remark 1.1.

(I) Some frequently used incidence rates satisfy assumption **(P1)**–**(P2)**, such as

(1) The bilinear incidence rates $f_1(S_h, I_v) = \beta_1 S_h I_v$ and $f_2(S_v, I_h) = \beta_2 S_v I_h$, $\beta_i > 0$, $i = 1, 2$ [4,32];

(2) The saturated incidence rates $f_1(S_h, I_v) = \frac{\beta_1 S_h I_v}{1+\varrho_1 I_v}$ and $f_2(S_v, I_h) = \frac{\beta_2 S_v I_h}{1+\varrho_2 I_h}$, $\beta_i, \varrho_i > 0$, $i = 1, 2$ [20];

(3) The saturated incidence rates $f_1(S_h, I_v) = \frac{\beta_1 S_h I_v}{1+\varrho_1 S_h}$ and $f_2(S_v, I_h) = \frac{\beta_2 S_v I_h}{1+\varrho_2 S_v}$, $\beta_i, \varrho_i > 0$, $i = 1, 2$; [27]

(4) The mixture of bilinear and saturated incidence rates $f_1(S_h, I_v) = \beta_1 S_h I_v$ and $f_2(S_v, I_h) = \frac{\beta_2 S_v I_h}{1+\varrho_2 S_v}$ or $f_1(S_h, I_v) = \frac{\beta_1 S_h I_v}{1+\varrho_1 I_v}$ and $f_2(S_v, I_h) = \beta_2 S_v I_h$, $\beta_i, \varrho_i > 0$, $i = 1, 2$;

(II) From [39], the capacity of activity for susceptible individuals is usually stronger than that of infected individuals. Thus, the assumption **(P3)** is reasonable.

The main purpose of this paper is to confirm the existence and nonexistence of nontrivial TWS for system (1.2). More specifically, we intend to state the main strategies of this work. By using the smallest positive root of the characteristic equation for the linearized system of (1.2), the suitable sub- and super-solutions are constructed, and so the existence of solutions for the auxiliary truncated system is obtained in view of Schauder's fixed-point theorem. Then, by means of limiting arguments and comparison principle, we obtain that system (1.2) admits a nontrivial bounded TWS connecting the disease-free equilibrium when $\mathfrak{R}_0 > 1$ and $c \geq c_*$. By establishing an appropriate Lyapunov functional and using LaSalle's invariance principle, it is proved that the TWS converges to the endemic equilibrium at positive infinity. Thanks to the detailed analysis, we obtain the nonexistence of nontrivial TWS when $\mathfrak{R}_0 \leq 1$ and $c > 0$ by contradiction. For the case of $\mathfrak{R}_0 > 1$ and $0 < c < c_*$, by proving that the I_h or I_v will change sign, we show that there is no nontrivial TWS connecting disease-free equilibrium and endemic equilibrium due to a contradiction. Our conclusions indicate that c_* is the minimal wave speed of system (1.2).

Some key improvements are necessary due to the introduce of general incidence and constant recruitment in this paper. To investigate the existence and boundedness of TWS, the author [36] assumed that the functions $g_2(\cdot)$ and $g_3(\cdot)$ are bounded [see (A5)]. However, this assumption does not apply to bilinear incidence. More precisely, if the f_1 and f_2 in model (1.2) are bilinear, then the methods of [36] cannot be employed to obtain the existence and boundedness of TWS for system (1.2). Hence, we need to utilize the ideas in [39] to overcome these technical difficulties to make our model cover more special cases. Actually, due to the introduce of general incidence, the mathematical analysis of the problem is more complicated and the results are more profound. It should be pointed out that, although it is an effective approach to deal with the nonexistence of TWS in the case of $\mathfrak{R}_0 > 1$ and $0 < c < c_*$ applying Laplace transform (e.g. [16,41]), this method is no longer applicable because it is difficult to verify the exponential decay of the solutions. Fortunately, we establish the nonexistence of this case with the help of an ingenious technique, which is easier to understand than the approach of Laplace transform.

The remainder of the paper is organized as follows. Section 2 presents some preliminaries which will be used in subsequent sections. Section 3 addresses the existence of TWS for system (1.2). Section 4 proves the nonexistence of TWS. Section 5 performs numerical simulations to verify the analytical results, and explores the sensitivity of minimal wave speed on parameters. Section 6 gives a brief discussion to conclude the article.

2. Preliminaries

To study the traveling wave solutions of (1.2), the constant equilibria are needed. It follows from **(P2)** that system (1.2) admits a disease-free equilibrium $E_0 = (S_h^0, 0, S_v^0, 0)^T$, where $S_h^0 := \Lambda/\mu_1$ and $S_v^0 := M/\mu_2$. To find a positive constant endemic equilibrium, we consider the following ODE (kinetic) system

$$\begin{cases} \mathrm{d}S_h(t)/\mathrm{d}t = \Lambda - f_1(S_h, I_v)(t) - \mu_1 S_h(t), \\ \mathrm{d}I_h(t)/\mathrm{d}t = f_1(S_h, I_v)(t) - \gamma_1 I_h(t), \\ \mathrm{d}S_v(t)/\mathrm{d}t = M - f_2(S_v, I_h)(t) - \mu_2 S_v(t), \\ \mathrm{d}I_v(t)/\mathrm{d}t = f_2(S_v, I_h)(t) - \gamma_2 I_v(t). \end{cases} \tag{2.1}$$

Epidemiology, the basic reproduction ratio $\mathfrak{R}_0$ is one of the most important concepts in infectious diseases, and it is a crucial threshold of disease outbreak or not [23]. According to [23], the $\mathfrak{R}_0$ of system (2.1) equals the spectral radius of the following matrix

$$\mathcal{M} := \begin{pmatrix} 0 & k_1/\gamma_1 \\ k_2/\gamma_2 & 0 \end{pmatrix},$$

and thus $\mathfrak{R}_0 = \sqrt{\bar{k}/\bar{\gamma}}$, where $\bar{k} := k_1 k_2$, $k_1 := \partial_{I_v} f_1(S_h^0, 0)$, $k_2 := \partial_{I_h} f_2(S_v^0, 0)$ and $\bar{\gamma} := \gamma_1\gamma_2$. Consider the following equations

$$\begin{cases} \Lambda - f_1(S_h^*, I_v^*) - \mu_1 S_h^* = 0, \\ f_1(S_h^*, I_v^*) - \gamma_1 I_h^* = 0, \\ M - f_2(S_v^*, I_h^*) - \mu_2 S_v^* = 0, \\ f_2(S_v^*, I_h^*) - \gamma_2 I_v^* = 0. \end{cases}$$

Adding the first two equations and last two equations of above system, respectively, we obtain $S_h^* = (\Lambda - \gamma_1 I_h^*)/\mu_1$ and $S_v^* = (M - \gamma_2 I_v^*)/\mu_2$. Obviously, $S_h^* > 0$ when $I_h^* \in (0, \Lambda/\gamma_1)$ and $S_v^* > 0$ when $I_v^* \in (0, M/\gamma_2)$. Substituting S_h^* and S_v^* into the second and fourth equations of above system, it follows that

$$\begin{cases} Q_1(I_h^*, I_v^*) = 0, \\ Q_2(I_h^*, I_v^*) = 0, \end{cases}$$

where $Q_1(I_h^*, I_v^*) := f_1((\Lambda - \gamma_1 I_h^*)/\mu_1, I_v^*) - \gamma_1 I_h^*$ and $Q_2(I_h^*, I_v^*) := f_2((M - \gamma_2 I_v^*)/\mu_2, I_h^*) - \gamma_2 I_v^*$. Since $Q_i(0, 0) = 0$, $i = 1, 2$, utilizing the implicit function existence theorem and [37], the above system has a unique $(I_h^*, I_v^*)^T$ satisfies $I_h^* \in (0, \Lambda/\gamma_1)$ and $I_v^* \in (0, M/\gamma_2)$ when $\mathfrak{R}_0 > 1$. Thus, system (2.1) has a unique endemic equilibrium $E_1^* := (S_h^*, I_h^*, S_v^*, I_v^*)^T$ provided that $\mathfrak{R}_0 > 1$.

From the definition of TWS, a solution $(S_h(t,x), I_h(t,x), S_v(t,x), I_v(t,x))^T$ of (1.2) is called a *traveling wave solution* if it has the form $(S_h(z), I_h(z), S_v(z), I_v(z))^T$, $z = x + ct$, $t \geq 0$, $x \in \mathbb{R}$ and $c > 0$ represents the wave speed. Then we have the wave profile equations

$$\begin{cases} cS_h'(z) = D_S S_h''(z) + \Lambda - f_1(S_h, I_v)(z) - \mu_1 S_h(z), \\ cI_h'(z) = D_I I_h''(z) + f_1(S_h, I_v)(z) - \gamma_1 I_h(z), \\ cS_v'(z) = d_S S_v''(z) + M - f_2(S_v, I_h)(z) - \mu_2 S_v(z), \\ cI_v'(z) = d_I I_v''(z) + f_2(S_v, I_h)(z) - \gamma_2 I_v(z), \end{cases} \tag{2.2}$$

wherein $' = \mathrm{d}/\mathrm{d}z$ and $'' = \mathrm{d}^2/\mathrm{d}z^2$. Our main objective is to find a nontrivial solution $(S_h(\cdot), I_h(\cdot), S_v(\cdot), I_v(\cdot))^T$ of system (2.2) satisfying boundary conditions

$$(S_h(-\infty), I_h(-\infty), S_v(-\infty), I_v(-\infty))^T = (S_h^0, 0, S_v^0, 0)^T, \tag{2.3}$$

and

$$(S_h(+\infty), I_h(+\infty), S_v(+\infty), I_v(+\infty))^T = (S_h^*, I_h^*, S_v^*, I_v^*)^T. \tag{2.4}$$

Linearizing (2.2) at E_0, one obtains

$$\begin{cases} cI_h'(z) = D_I I_h''(z) + k_1 I_v(z) - \gamma_1 I_h(z), \\ cI_v'(z) = d_I I_v''(z) + k_2 I_h(z) - \gamma_2 I_v(z). \end{cases}$$

Plugging $(I_h(z), I_v(z))^T = (\chi_2, \chi_4)^T \mathrm{e}^{\zeta z}$ into above system, we get

$$\begin{cases} c\zeta\chi_2 = D_I \chi_2 \zeta^2 + k_1 \chi_4 - \gamma_1 \chi_2, \\ c\zeta\chi_4 = d_I \chi_4 \zeta^2 + k_2 \chi_2 - \gamma_2 \chi_4. \end{cases} \tag{2.5}$$

Hence, the characteristic equation for the linearized system of (2.2) is

$$\Sigma^c(\zeta) := U_2^c(\zeta) U_4^c(\zeta) - \bar{k} = 0,$$

where $U_2^c(\zeta) := D_I\zeta^2 - c\zeta - \gamma_1$ and $U_4^c(\zeta) := d_I\zeta^2 - c\zeta - \gamma_2$. Denoting $\zeta_m^* := \min\{\zeta_1^+, \zeta_2^+\}$, where ζ_1^+ and ζ_2^+ are the positive roots of $U_2^c(\zeta) = 0$ and $U_4^c(\zeta) = 0$. Similar to the arguments of [26, Lemma 2.1], one can prove the following lemma.

Lemma 2.1. *Suppose $\mathfrak{R}_0 > 1$. Then there exist constants $c_* > 0$ and $\zeta^* > 0$ such that*

$$\left.\frac{\partial \Sigma^c(\zeta)}{\partial \zeta}\right|_{(c_*,\zeta^*)} = 0 \quad \text{and} \quad \Sigma^{c_*}(\zeta^*) = 0.$$

Furthermore,

(1) *If $0 < c < c_*$, then $\Sigma^c(\zeta) < 0$, for all $\zeta \in [0, \zeta_m^*)$;*
(2) *If $c > c_*$, then there are two positive roots $\zeta_1 := \zeta_{m1}(c)$ and $\zeta_2 := \zeta_{m2}(c)$ of $\Sigma^c(\zeta) = 0$, satisfying $\zeta_1 < \zeta^* < \zeta_2 < \zeta_m^*$, $\zeta_1'(c) < 0$, $\zeta_2'(c) > 0$, such that $U_j^c(\zeta_i) < 0$ $(i = 1, 2, j = 2, 4)$ and*

$$\Sigma^c(\zeta) = \begin{cases} < 0, & \lambda \in [0, \zeta_1) \cup (\zeta_2, \zeta_2 + \sigma), \\ > 0, & \lambda \in (\zeta_1, \zeta_2), \end{cases}$$

here $' = \mathrm{d}/\mathrm{d}c$ and $\sigma > 0$ is a sufficiently small constant. Moreover, there exist constants $\chi_2 = k_1$ and $\chi_4 = -U_2^c(\zeta_1)$ such that (2.5) holds for $\zeta = \zeta_1$.

2.1. Sub- and super-solutions

In the following, we always assume $\mathfrak{R}_0 > 1$ and fix $c > c_*$. To prove the existence of TWS, it is necessary to construct a pair of sub- and super-solutions. For $z \in \mathbb{R}$, define

$$\begin{aligned} &S_h^+(z) := S_h^0, && S_h^-(z) := \max\{S_h^0(1 - M_1 \mathrm{e}^{\epsilon_1 z}), 0\}, \\ &I_h^+(z) := \chi_2 \mathrm{e}^{\zeta_1 z}, && I_h^-(z) := \max\{\chi_2 \mathrm{e}^{\zeta_1 z}(1 - \mathcal{H} M_2 \mathrm{e}^{\epsilon_2 z}), 0\}, \\ &S_v^+(z) := S_v^0, && S_v^-(z) := \max\{S_v^0(1 - M_3 \mathrm{e}^{\epsilon_3 z}), 0\}, \\ &I_v^+(z) := \chi_4 \mathrm{e}^{\zeta_1 z}, && I_v^-(z) := \max\{\chi_4 \mathrm{e}^{\zeta_1 z}(1 - \mathcal{H} M_4 \mathrm{e}^{\epsilon_2 z}), 0\}, \end{aligned}$$

where χ_2, χ_4 and ζ_1 have been determined by Lemma 2.1, the $\mathcal{H}$, M_i $(i = 1, 2, 3, 4)$ and ϵ_j $(j = 1, 2, 3)$ will be chosen later.

Lemma 2.2. *The functions $S_h^+(z) = S_h^0$ and $S_v^+(z) = S_v^0$ satisfy*

$$\Lambda - f_1(S_h^+, I_v^-)(z) - \mu_1 S_h^+(z) \le 0, \quad M - f_2(S_v^+, I_h^-)(z) - \mu_2 S_v^+(z) \le 0, \quad z \in \mathbb{R}.$$

Proof. The proof is obvious and so omitted. □

Lemma 2.3. *The functions $I_h^+(z) = \chi_2 \mathrm{e}^{\zeta_1 z}$ and $I_v^+(z) = \chi_4 \mathrm{e}^{\zeta_1 z}$ satisfy*

$$D_I {I_h^+}''(z) - c{I_h^+}'(z) - \gamma_1 I_h^+(z) + f_1(S_h^0, I_v^+)(z) \le 0,$$

and

$$d_I {I_v^+}''(z) - c{I_v^+}'(z) - \gamma_2 I_v^+(z) + f_2(S_v^0, I_h^+)(z) \le 0, \quad z \in \mathbb{R}.$$

Proof. By **(P1)** and **(P2)**, mean value theorem yields that

$$f_1(S_h^0, I_v)(z) \le \partial_{I_v} f_1(S_h^0, 0) I_v(z) = k_1 I_v(z), \tag{2.6}$$

and

$$f_2(S_v^0, I_h)(z) \le \partial_{I_h} f_2(S_v^0, 0) I_h(z) = k_2 I_h(z), \tag{2.7}$$

for all $z \in \mathbb{R}$. Then one has

$$D_I {I_h^+}''(z) - c{I_h^+}'(z) - \gamma_1 I_h^+(z) + f_1(S_h^0, I_v^+)(z)$$

$$\begin{aligned}&\leq D_I I_h^{+''}(z) - cI_h^{+'}(z) - \gamma_1 I_h^+(z) + k_1 I_v^+(z)\\&= \mathrm{e}^{\zeta_1 z}[\chi_2(D_I\zeta_1^2 - c\zeta_1 - \gamma_1) + k_1\chi_4]\\&= \mathrm{e}^{\zeta_1 z}[\chi_2 U_2^c(\zeta_1) + k_1\chi_4]\\&= 0,\end{aligned}$$

and

$$d_I I_v^{+''}(z) - cI_v^{+'}(z) - \gamma_2 I_v^+(z) + f_2(S_v^0, I_h^+)(z) \leq \mathrm{e}^{\zeta_1 z}[\chi_4 U_4^c(\zeta_1) + k_2\chi_2] = 0.$$

This ends the proof. □

Lemma 2.4. *Suppose*

$$0 < \epsilon_1 < \min\left\{\zeta_1, \frac{c}{D_S}\right\}, \;\; M_1 > \max\left\{1, \frac{k_1\chi_4}{S_h^0(-D_S\epsilon_1^2 + c\epsilon_1 + \mu_1)}\right\},$$

and

$$0 < \epsilon_3 < \min\left\{\zeta_1, \frac{c}{d_S}\right\}, \;\; M_3 > \max\left\{1, \frac{k_2\chi_2}{S_v^0(-d_S\epsilon_1^2 + c\epsilon_1 + \mu_2)}\right\}.$$

Then $S_h^-(z) = \max\{S_h^0(1 - M_1\mathrm{e}^{\epsilon_1 z}), 0\}$ *and* $S_v^-(z) = \max\{S_v^0(1 - M_3\mathrm{e}^{\epsilon_3 z}), 0\}$ *satisfy*

$$D_S S_h^{-''}(z) - cS_h^{-''}(z) + \Lambda - f_1(S_h^-, I_v^+)(z) - \mu_1 S_h^-(z) \geq 0, \;\; z \neq z_1 := -\frac{\ln M_1}{\epsilon_1}, \tag{2.8}$$

and

$$d_S S_v^{-''}(z) - cS_v^{-''}(z) + M - f_2(S_v^-, I_h^+)(z) - \mu_2 S_v^-(z) \geq 0, \;\; z \neq z_3 := -\frac{\ln M_3}{\epsilon_3}. \tag{2.9}$$

Proof. As $z > z_1$, it is clear that (2.8) holds due to $S_h^-(z) = 0$. As $z < z_1$, we get $S_h^-(z) = S_h^0(1 - M_1\mathrm{e}^{\epsilon_1 z})$. Following from (2.6) that

$$\begin{aligned}&D_S S_h^{-''}(z) - cS_h^{-'}(z) + \Lambda - f_1(S_h^-, I_v^+)(z) - \mu_1 S_h^-(z)\\&\quad\geq D_S S_h^{-''}(z) - cS_h^{-'}(z) + \Lambda - \mu_1 S_h^-(z) - k_1 I_v^+(z)\\&\quad= -D_S M_1 S_h^0\epsilon_1^2\mathrm{e}^{\epsilon_1 z} + cM_1 S_h^0\epsilon_1\mathrm{e}^{\epsilon_1 z} + \Lambda - \mu_1 S_h^0 + \mu_1 M_1 S_h^0\mathrm{e}^{\epsilon_1 z} - k_1\chi_4\mathrm{e}^{\zeta_1 z}\\&\quad\geq \mathrm{e}^{\epsilon_1 z}[M_1 S_h^0(-D_S\epsilon_1^2 + c\epsilon_1 + \mu_1) - k_1\chi_4].\end{aligned}$$

Then (2.8) holds by the condition for M_1. In the similar fashion, one can show that (2.9) is true for S_v^- by using (2.7). This completes the proof. □

Lemma 2.5. *Assume* $0 < 2\epsilon_2 < \min\{\zeta_1, \epsilon_1, \epsilon_3\}$. *Then there exists sufficiently large* $\mathcal{H} > 0$ *such that* $I_h^-(z) = \max\{\chi_2\mathrm{e}^{\zeta_1 z}(1 - \mathcal{H}M_2\mathrm{e}^{\epsilon_2 z}), 0\}$ *and* $I_v^-(z) = \max\{\chi_4\mathrm{e}^{\zeta_1 z}(1 - \mathcal{H}M_4\mathrm{e}^{\epsilon_2 z}), 0\}$ *satisfy*

$$D_I I_h^{-''}(z) - cI_h^{-'}(z) - \gamma_1 I_h^-(z) + f_1(S_h^-, I_v^-)(z) \geq 0, \quad z \neq z_2 := -\frac{\ln(\mathcal{H}M_2)}{\epsilon_2}, \tag{2.10}$$

and

$$d_I I_v^{-''}(z) - cI_v^{-'}(z) - \gamma_2 I_v^-(z) + f_2(S_v^-, I_h^-)(z) \geq 0, \quad z \neq z_4 := -\frac{\ln(\mathcal{H}M_4)}{\epsilon_2}, \tag{2.11}$$

where $\mathcal{H}$ *meets* $\max\{z_2, z_4\} < \min\{z_1, z_3\}$.

Proof. Without loss of generality, assuming $z_2 < z_4$. It is not difficult to see that (2.10) holds for $z > z_2$ and (2.11) holds for $z > z_4$. Since $z_4 < \min\{z_1, z_3\}$, we have

$$I_h^-(z) \geq \chi_2\mathrm{e}^{\zeta_1 z}(1 - \mathcal{H}M_2\mathrm{e}^{\epsilon_2 z}), \;\; I_v^-(z) = \chi_4\mathrm{e}^{\zeta_1 z}(1 - \mathcal{H}M_4\mathrm{e}^{\epsilon_2 z}), \;\; S_v^-(z) = S_v^0(1 - M_3\mathrm{e}^{\epsilon_3 z}), \;\; z < z_4.$$

When $z < z_4$, one has

$$d_I I_v^{-''}(z) - cI_v^{-'}(z) - \gamma_2 I_v^-(z) + f_2(S_v^-, I_h^-)(z)$$

$$
\begin{aligned}
&= d_I[\chi_4\zeta_1^2 e^{\zeta_1 z} - \chi_4 M_4 \mathcal{H}(\zeta_1+\epsilon_2)^2 e^{(\zeta_1+\epsilon_2)z}] - c[\chi_4\zeta_1 e^{\zeta_1 z} - \chi_4 M_4\mathcal{H}(\zeta_1+\epsilon_2)e^{(\zeta_1+\epsilon_2)z}] \\
&\quad - \gamma_2[\chi_4 e^{\zeta_1 z} - \chi_4 M_4 \mathcal{H} e^{(\zeta_1+\epsilon_2)z}] + f_2(S_v^-, I_h^-)(z) \\
&= e^{\zeta_1 z}\chi_4(d_I\zeta_1^2 - c\zeta_1 - \gamma_2) + \chi_4 M_4\mathcal{H}e^{(\zeta_1+\epsilon_2)z}[-d_I(\zeta_1+\epsilon_2)^2 + c(\zeta_1+\epsilon_2)+\gamma_2] \\
&\quad + f_2(S_v^-, I_h^-)(z) \\
&= e^{\zeta_1 z}\chi_4 U_4^c(\zeta_1) - e^{(\zeta_1+\epsilon_2)z}\chi_4 M_4\mathcal{H}U_4^c(\zeta_1+\epsilon_2) + f_2(S_v^-, I_h^-)(z).
\end{aligned}
$$

Since $\chi_2 k_2 + \chi_4 U_4^c(\zeta_1) = 0$, to prove (2.11) for $z < z_4$, it is sufficient to show

$$
-e^{\zeta_1 z}\chi_2 k_2 - e^{(\zeta_1+\epsilon_2)z}\chi_4 M_4\mathcal{H}U_4^c(\zeta_1+\epsilon_2) + f_2(S_v^-, I_h^-)(z) \ge 0. \tag{2.12}
$$

Because $I_h^-(z) \ge \chi_2 e^{\zeta_1 z}(1 - \mathcal{H}M_2 e^{\epsilon_2 z})$, $z \in \mathbb{R}$ and $k_2 = \partial_{I_h} f_2(S_v^0, 0)$, one gets

$$
\begin{aligned}
&f_2(S_v^-, I_h^-)(z) - e^{\zeta_1 z}\chi_2 k_2 \\
&\quad = f_2(S_v^-, I_h^-)(z) - e^{\zeta_1 z}\chi_2\partial_{I_h} f_2(S_v^0, 0) \\
&\quad = f_2(S_v^-, I_h^-)(z) - \partial_{I_h} f_2(S_v^0,0)I_h^-(z) + \partial_{I_h} f_2(S_v^0,0)I_h^-(z) - e^{\zeta_1 z}\chi_2\partial_{I_h} f_2(S_v^0,0) \\
&\quad \ge f_2(S_v^-, I_h^-)(z) - \partial_{I_h} f_2(S_v^0,0)I_h^-(z) + \partial_{I_h} f_2(S_v^0,0)[\chi_2 e^{\zeta_1 z} - \chi_2\mathcal{H}M_2 e^{(\zeta_1+\epsilon_2)z}] \\
&\qquad - e^{\zeta_1 z}\chi_2\partial_{I_h} f_2(S_v^0,0) \\
&\quad = f_2(S_v^-, I_h^-)(z) - \partial_{I_h} f_2(S_v^0,0)I_h^-(z) - \chi_2 M_2\mathcal{H}\partial_{I_h} f_2(S_v^0,0)e^{(\zeta_1+\epsilon_2)z}.
\end{aligned}
$$

Thus, to verify (2.12), we only to show

$$
-e^{(\zeta_1+\epsilon_2)z}\chi_4 M_4\mathcal{H}U_4^c(\zeta_1+\epsilon_2) - e^{(\zeta_1+\epsilon_2)z}\chi_2 M_2\mathcal{H}\partial_{I_h} f_2(S_v^0,0) + f_2(S_v^-, I_h^-)(z) - \partial_{I_h} f_2(S_v^0,0)I_h^-(z) \ge 0. \tag{2.13}
$$

By appealing to Taylor's theorem (see [2, Sect. 5.5]) and assumptions **(P1)**–**(P2)**, we have

$$
\begin{aligned}
f_2(S_v^-, I_h^-) &= \partial_{I_h} f_2(S_v^-, \xi_{I_h^-})I_h^- \\
&= [\partial_{I_h} f_2(S_v^0,0) + \partial_{S_v}\partial_{I_h} f_2(\xi_{S_v^-}, \xi_{I_h^-})(S_v^- - S_v^0) + \partial_{I_h}^2 f_2(S_v^-, \tilde{\xi}_{I_h^-})\xi_{I_h^-}]I_h^- \\
&\ge [\partial_{I_h} f_2(S_v^0,0) + \partial_{S_v}\partial_{I_h} f_2(\xi_{S_v^-}, \xi_{I_h^-})(S_v^- - S_v^0) + \partial_{I_h}^2 f_2(S_v^-, \tilde{\xi}_{I_h^-})I_h^-]I_h^-,
\end{aligned}
$$

where $0 \le \xi_{S_v^-} \le S_v^- \le S_v^+ = S_v^0$, $0 \le \tilde{\xi}_{I_h^-} \le \xi_{I_h^-} \le I_h^- \le I_h^+ = \chi_2 e^{\zeta_1 z}$, $z < z_4 < 0$. Therefore,

$$
f_2(S_v^-, I_h^-) - \partial_{I_h} f_2(S_v^0,0)I_h^- \ge -e^{\epsilon_3 z}M_3\partial_{S_v}\partial_{I_h} f_2(\xi_{S_v^-}, \xi_{I_h^-})I_h^- + \partial_{I_h}^2 f_2(S_v^-, \tilde{\xi}_{I_h^-})(I_h^-)^2.
$$

Since $0 \le \xi_{S_v^-} \le S_v^0$, $0 \le \tilde{\xi}_{I_h^-} \le \xi_{I_h^-} \le \chi_2 e^{\zeta_1 z}$, $z < 0$, there exists a constant $C_1 > 0$ such that

$$
\left|\partial_{S_v}\partial_{I_h} f_2(\xi_{S_v^-}, \xi_{I_h^-})\right| + \left|\partial_{I_h}^2 f_2(S_v^-, \tilde{\xi}_{I_h^-})\right| \le C_1.
$$

Then

$$
f_2(S_v^-, I_h^-) - \partial_{I_h} f_2(S_v^0,0)I_h^- \ge -e^{\epsilon_3 z}C_1 M_3 I_h^- - C_1(I_h^-)^2.
$$

Owing to $0 \le I_h^- \le I_h^+ = \chi_2 e^{\zeta_1 z}$, we obtain

$$
f_2(S_v^-, I_h^-) - \partial_{I_h} f_2(S_v^0,0)I_h^- \ge -e^{(\zeta_1+\epsilon_3)z}C_1\chi_2 M_3 - C_1\chi_2^2 e^{2\zeta_1 z}.
$$

So, to prove (2.13), it is enough to show

$$
-\mathcal{H}[\chi_4 M_4 U_4^c(\zeta_1+\epsilon_2) + \chi_2 M_2 k_2] - e^{(\epsilon_3-\epsilon_2)z}C_1(\chi_2 M_3 + \chi_2^2) \ge 0 \tag{2.14}
$$

which is owing to $0 < \epsilon_3 < \zeta_1$ from Lemma 2.4. Similarly, to show (2.10) for $z < z_2$, since $0 < \epsilon_1 < \zeta_1$ by Lemma 2.4, one needs to prove

$$
-\mathcal{H}[\chi_2 M_2 U_2^c(\zeta_1+\epsilon_2) + \chi_4 M_4 k_1] - e^{(\epsilon_1-\epsilon_2)z}C_2(\chi_4 M_1 + \chi_4^2) \ge 0, \tag{2.15}
$$

for some $C_2 > 0$. According to [36, Lemma 2.4], there are two positive constants M_2, M_4 such that for $\zeta_1 + \epsilon_2 < \zeta_2 < \zeta_m^*$, we have

$$\begin{cases} M_2\chi_2 U_2^c(\zeta_1+\epsilon_2) + M_4\chi_4 k_1 < 0, \\ M_4\chi_4 U_4^c(\zeta_1+\epsilon_2) + M_2\chi_2 k_2 < 0. \end{cases}$$

Let

$$h_1(\epsilon_2) := M_2\chi_2 U_2^c(\zeta_1+\epsilon_2) + M_4\chi_4 k_1, \;\; h_2(\epsilon_2) := M_4\chi_4 U_4^c(\zeta_1+\epsilon_2) + M_2\chi_2 k_2$$

Choose $\mathcal{H}$ to be large enough satisfying

$$\mathcal{H} > \max\left\{\frac{C_1(\chi_2 M_3 + \chi_2^2)}{-h_2(\epsilon_2)}, \; \frac{C_2(\chi_4 M_1 + \chi_4^2)}{-h_1(\epsilon_2)}\right\}.$$

Then, when $z < z_4 < 0$, one has

$$\begin{aligned} &-\mathcal{H}[\chi_4 M_4 U_4^c(\zeta_1+\epsilon_2) + \chi_2 M_2 k_2] - \mathrm{e}^{(\epsilon_3-\epsilon_2)z} C_1(\chi_2 M_3 + \chi_2^2) \\ &> \frac{C_1(\chi_2 M_3 + \chi_2^2)}{-h_2(\epsilon_2)} \cdot [-h_2(\epsilon_2)] - C_1(\chi_2 M_3 + \chi_2^2) \\ &= 0 \end{aligned}$$

which is due to $-h_2(\epsilon_2) > 0$, $\epsilon_3 - \epsilon_2 > 0$. When $z < z_2 < 0$, one obtains

$$\begin{aligned} &-\mathcal{H}[\chi_2 M_2 U_2^c(\zeta_1+\epsilon_2) + \chi_4 M_4 k_1] - \mathrm{e}^{(\epsilon_1-\epsilon_2)z} C_2(\chi_4 M_1 + \chi_4^2) \\ &> \frac{C_2(\chi_4 M_1 + \chi_4^2)}{-h_1(\epsilon_2)} \cdot [-h_1(\epsilon_2)] - C_2(\chi_4 M_1 + \chi_4^2) \\ &= 0 \end{aligned}$$

which is owing to $-h_1(\epsilon_2) > 0$, $\epsilon_1 - \epsilon_2 > 0$. Therefore, (2.10) and (2.11) hold when $\mathcal{H}$ satisfies the above inequalities such that $\max\{z_2, z_4\} < \min\{z_1, z_3\}$. This ends the proof. □

2.2. An auxiliary truncated problem

Let $X > \max\{-z_2, -z_4\}$. Then define

$$\Gamma_X := \left\{ (\phi_1,\phi_2,\phi_3,\phi_4)^T \in C([-X,X],\mathbb{R}^4) \;\middle|\; \begin{array}{l} \phi_1(\pm X) = S_h^-(\pm X),\, S_h^-(z) \le \phi_1(z) \le S_h^0, \\ \phi_2(\pm X) = I_h^-(\pm X),\, I_h^-(z) \le \phi_2(z) \le I_h^+(z), \\ \phi_3(\pm X) = S_v^-(\pm X),\, S_v^-(z) \le \phi_3(z) \le S_v^0, \\ \phi_4(\pm X) = I_v^-(\pm X),\, I_v^-(z) \le \phi_4(z) \le I_v^+(z), \\ \forall z \in [-X,X]. \end{array} \right\}.$$

and

$$\tilde{\phi}_1(z) = \begin{cases} \phi_1(z), & |z| \le X, \\ S_h^-(z), & |z| > X, \end{cases} \qquad \tilde{\phi}_2(z) = \begin{cases} \phi_2(z), & |z| \le X, \\ I_h^-(z), & |z| > X, \end{cases}$$

and

$$\tilde{\phi}_3(z) = \begin{cases} \phi_3(z), & |z| \le X, \\ S_v^-(z), & |z| > X, \end{cases} \qquad \tilde{\phi}_4(z) = \begin{cases} \phi_4(z), & |z| \le X, \\ I_v^-(z), & |z| > X. \end{cases}$$

Thus, it is easy to see that the set Γ_X is a bounded closed convex set.

For $z \in (-X, X)$, consider the following boundary-value problem

$$\begin{cases} cS'_{h,X}(z) = D_S S''_{h,X}(z) + \Lambda - f_1(S_{h,X}, \phi_4)(z) - \mu_1 S_{h,X}(z), \\ cI'_{h,X}(z) = D_I I''_{h,X}(z) + f_1(\tilde{\phi}_1, \tilde{\phi}_4)(z) - \gamma_1 I_{h,X}(z), \\ cS'_{v,X}(z) = d_S S''_{v,X}(z) + M - f_2(S_{v,X}, \phi_2)(z) - \mu_2 S_{v,X}(z), \\ cI'_{v,X}(z) = d_I I''_{v,X}(z) + f_2(\tilde{\phi}_3, \tilde{\phi}_2)(z) - \gamma_2 I_{v,X}(z), \end{cases} \tag{2.16}$$

satisfying boundary conditions

$$S_{h,X}(\pm X) = S_h^-(\pm X),\ I_{h,X}(\pm X) = I_h^-(\pm X),\ S_{v,X}(\pm X) = S_v^-(\pm X),\ I_{v,X}(\pm X) = I_v^-(\pm X). \tag{2.17}$$

According to the standard ODE theory, problems (2.16)–(2.17) admit a unique solution $(S_{h,X}(z), I_{h,X}(z), S_{v,X}(z), I_{v,X}(z))^T$ satisfying $S_{h,X},\ I_{h,X},\ S_{v,X}$ and $I_{v,X} \in W_p^2((-X, X), \mathbb{R}) \cap C([-X, X],\ \mathbb{R})$, for any $p \in \mathbb{N}^*$ (the set of positive integers) (see [10, Corollary 9.18]). Furthermore, by using the embedding theorem [10, Theorem 7.26], we know that $S_{h,X},\ I_{h,X},\ S_{v,X}$ and $I_{v,X} \in W_p^2((-X, X), \mathbb{R}) \to C^{1,\nu}([-X, X])$, $\nu \in (0, 1)$. Define an operator $\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)^T$ on Γ_X as follows

$$S_{h,X} = \mathcal{F}_1(\phi_1, \phi_2, \phi_3, \phi_4),\ \ I_{h,X} = \mathcal{F}_2(\phi_1, \phi_2, \phi_3, \phi_4),$$

and

$$S_{v,X} = \mathcal{F}_3(\phi_1, \phi_2, \phi_3, \phi_4),\ \ I_{v,X} = \mathcal{F}_4(\phi_1, \phi_2, \phi_3, \phi_4),$$

for all $(\phi_1, \phi_2, \phi_3, \phi_4)^T \in \Gamma_X$.

Lemma 2.6. *The operator $\mathcal{F}$ maps Γ_X into Γ_X, i.e., $\mathcal{F}(\Gamma_X) \subset \Gamma_X$.*

Proof. It is obvious that 0 is the sub-solution of the first and third equations for system (2.16) on $(-X, X)$, and S_h^0 and S_v^0 are the super-solutions of the first and third equations for system (2.16) on $(-X, X)$.

Since $0 = S_{h,X}(X) = S_h^-(X) < S_h^0$ and $0 < S_{h,X}(-X) = S_h^-(-X) < S_h^0$, it follows from the maximum principle that $0 \le S_{h,X}(z) \le S_h^0$, $z \in [-X, X]$. Recalling that $\max\{z_2, z_4\} < \min\{z_1, z_3\}$ and $X > \max\{-z_2, -z_4\}$, by (2.8) and assumption **(P1)**, we get

$$\begin{aligned} 0 &\le D_S {S_h^-}''(z) - c{S_h^-}'(z) + \Lambda - f_1(S_h^-, I_v^+)(z) - \mu_1 S_h^-(z) \\ &\le D_S {S_h^-}''(z) - c{S_h^-}'(z) + \Lambda - f_1(S_h^-, \phi_4)(z) - \mu_1 S_h^-(z) \end{aligned}$$

in $[-X, z_1]$. By the maximum principle and the facts $S_{h,X}(-X) = S_h^-(-X)$ and $S_{h,X}(z_1) \ge S_h^-(z_1) = 0$, one has $S_h^-(z) \le S_{h,X}(z)$, for any $[-X, z_1]$. In addition, $0 = S_h^-(z) \le S_{h,X}(z)$ in $[z_1, X]$. Consequently, $S_h^-(z) \le S_{h,X}(z) \le S_h^0$, $z \in [-X, X]$. Similar discussions can be showed $S_v^-(z) \le S_{v,X}(z) \le S_v^0$, $z \in [-X, X]$.

Next to consider $I_{h,X}(z)$ and $I_{v,X}(z)$. It is clear that 0 is the sub-solution of the second and fourth equations for system (2.16) on $[-X, X]$. Since $\tilde{\phi}_1(z) \le S_h^0$ and $\tilde{\phi}_4(z) \le I_v^+(z)$, $z \in [-X, X]$, following from **(P1)** and Lemma 2.3 that

$$D_I {I_h^+}''(z) - c{I_h^+}'(z) - \gamma_1 I_h^+(z) + f_1(\tilde{\phi}_1, \tilde{\phi}_4)(z) \le D_I {I_h^+}''(z) - c{I_h^+}'(z) - \gamma_1 I_h^+(z) + f_1(S_h^0, I_v^+)(z) \le 0,$$

for any $z \in [-X, X]$. Thus, $I_h^+(z)$ is the super-solution of the second equation for (2.16) in $z \in [-X, X]$. Moreover, by $\tilde{\phi}_1(z) \ge S_h^-(z)$ and $\tilde{\phi}_4(z) \ge I_v^-(z)$, $z \in [-X, X]$, combining **(P1)** and Lemma 2.5 that

$$D_I {I_h^-}''(z) - c{I_h^-}'(z) - \gamma_1 I_h^-(z) + f_1(\tilde{\phi}_1, \tilde{\phi}_4)(z) \ge D_I {I_h^-}''(z) - c{I_h^-}'(z) - \gamma_1 I_h^-(z) + f_1(S_h^-, I_v^-)(z) \ge 0,$$

for $z \in [-X, X]$. So, $I_h^-(z)$ is the sub-solution of the second equation for (2.16) in $z \in [-X, X]$. Accordingly, $I_h^-(z) \le I_{h,X}(z) \le I_h^+(z)$, $z \in [-X, X]$. In the similar way, $I_v^-(z) \le I_{v,X}(z) \le I_v^+(z)$, $z \in [-X, X]$. This completes the proof. □

Lemma 2.7. *The operator $\mathcal{F}\colon \Gamma_X \to \Gamma_X$ is completely continuous.*

Proof. To prove the compactness of $\mathcal{F}$. Suppose $(S_{h,X}(z), I_{h,X}(z), S_{v,X}(z), I_{v,X}(z))^T$ is the solution of problems (2.16)–(2.17). Then the first and second derivatives of $(S_{h,X}(z), I_{h,X}(z), S_{v,X}\ (z), I_{v,X}(z))$ with respect to z are bounded on $[-X, X]$ according to embedding theorem. Hence, Arzelà-Ascoli theorem yields that $\mathcal{F}$ is compact.

To show the continuity of $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)^T$. For $(\phi_1^1(\cdot), \phi_2^1(\cdot), \phi_3^1(\cdot), \phi_4^1(\cdot))^T \in \Gamma_X$ and $(\phi_1^2(\cdot), \phi_2^2(\cdot), \phi_3^2(\cdot), \phi_4^2(\cdot))^T \in \Gamma_X$, setting

$$S_{h,X}^j = \mathcal{F}_1(\phi_1^j, \phi_2^j, \phi_3^j, \phi_4^j), \quad j = 1, 2.$$

For the operator $\mathcal{F}_1$. By direct calculations, we have

$$\begin{aligned} &D_S[S_{h,X}^1 - S_{h,X}^2]''(z) - c[S_{h,X}^1 - S_{h,X}^2]'(z) - \mu_1[S_{h,X}^1 - S_{h,X}^2](z) \\ &\quad = [f_1(S_{h,X}^1, \phi_4^1)(z) - f_1(S_{h,X}^2, \phi_4^2)(z)] \\ &\quad \le L_1|S_{h,X}^1(z) - S_{h,X}^2(z)| + k_1|\phi_4^1(z) - \phi_4^2(z)|, \end{aligned}$$

where

$$L_1 := \max_{0 \le S_h(z) \le S_h^0,\, z \in [-X,X]} \partial_{S_h} f_1(S_h, I_v^+), \qquad k_1 = \partial_{I_v} f_1(S_h^0, 0).$$

Thus, the globally elliptic estimate and embedding theorem give that $\mathcal{F}_1$ is continuous. Similarly, we can show the continuity of $\mathcal{F}_i$ $(i = 2, 3, 4)$. This ends the proof. □

Combining Lemmas 2.6 and 2.7, Schauder's fixed-point theorem implies that there exists $(S_{h,X}, I_{h,X}, S_{v,X}, I_{v,X})^T \in \Gamma_X$ satisfying

$$(S_{h,X}, I_{h,X}, S_{v,X}, I_{v,X})^T = \mathcal{F}(S_{h,X}, I_{h,X}, S_{v,X}, I_{v,X}), \;\; z \in [-X, X].$$

Then $(S_{h,X}, I_{h,X}, S_{v,X}, I_{v,X})^T$ satisfies

$$\begin{cases} cS_{h,X}'(z) = D_S S_{h,X}''(z) + \Lambda - f_1(S_{h,X}, I_{v,X})(z) - \mu_1 S_{h,X}(z), \\ cI_{h,X}'(z) = D_I I_{h,X}''(z) + f_1(\tilde{S}_{h,X}, \tilde{I}_{v,X})(z) - \gamma_1 I_{h,X}(z), \\ cS_{v,X}'(z) = d_S S_{v,X}''(z) + M - f_2(S_{v,X}, I_{h,X})(z) - \mu_2 S_{v,X}(z), \\ cI_{v,X}'(z) = d_I I_{v,X}''(z) + f_2(\tilde{S}_{v,X}, \tilde{I}_{h,X})(z) - \gamma_2 I_{v,X}(z), \end{cases} \tag{2.18}$$

for $z \in (-X, X)$, wherein

$$\tilde{S}_{h,X}(z) = \begin{cases} S_{h,X}(z), & |z| \le X, \\ S_h^-(z), & |z| > X, \end{cases} \quad \tilde{I}_{h,X}(z) = \begin{cases} I_{h,X}(z), & |z| \le X, \\ I_h^-(z), & |z| > X, \end{cases} \tag{2.19}$$

and

$$\tilde{S}_{v,X}(z) = \begin{cases} S_{v,X}(z), & |z| \le X, \\ S_v^-(z), & |z| > X, \end{cases} \quad \tilde{I}_{v,X}(z) = \begin{cases} I_{v,X}(z), & |z| \le X, \\ I_v^-(z), & |z| > X. \end{cases} \tag{2.20}$$

Define

$$C^{2,1}([-X, X]) := \{u \in C^2([-X, X]) \,|\, u, u' \text{ and } u'' \text{ are Lipschitz continuous}\}$$

with the norm

$$\|u\|_{C^{2,1}([-X,X])} = \max_{z \in [-X,X]} |u| + \max_{z \in [-X,X]} |u'| + \max_{z \in [-X,X]} |u''| + \sup_{\substack{z,\, y \in [-X,X] \\ z \ne y}} \frac{|u''(z) - u''(y)|}{|z - y|}.$$

Then there are the following estimates for $S_{h,X}$, $I_{h,X}$, $S_{v,X}$, and $I_{v,X}$.

Lemma 2.8. *For given $Y > 0$, there exists a constant $P := P(Y) > 0$ such that*

$$\|S_{h,X}\|_{C^3[-Y,Y]},\ \|S_{v,X}\|_{C^3[-Y,Y]},\ \|I_{h,X}\|_{C^{2,1}[-Y,Y]},\ \|I_{v,X}\|_{C^{2,1}[-Y,Y]} \le P,$$

for $0 < Y < X$ and $X > \max\{-z_2, -z_4\}$.

Proof. Since $S_{h,X}(z) \le S_h^0$ and $I_{v,X}(z) \le \chi_4 e^{\zeta_1 z}$, $z \in [-Y, Y]$, utilizing the L^p $(p \ge 2)$ estimates [39] of linear elliptic differential equations to the first equation of system (2.18) yields that

$$\|S_{h,X}\|_{W_p^2(-Y,Y)} \le C[\Lambda + f_1(S_h^0, \chi_4 e^{\zeta_1 z}) + \|\eta_1\|_{W_p^2(-Y,Y)}],$$

where $C := C(Y) > 0$ and η_1 is chosen to be a linear function connecting the points $(-Y, S_{h,X}\ (-Y))$ and $(Y, S_{h,X}(Y))$. So, there is a constant $\bar{P} := \bar{P}(Y) > 0$ such that $\|S_{h,X}\|_{W_p^2(-Y,Y)} \le \bar{P}$ for all $X > Y$. Furthermore, it follows from the fact $W_p^2(-Y,Y) \hookrightarrow C^{1,\nu}[-Y,Y]$, $\nu = 1 - 1/p$ that there exists a $\tilde{P} := \tilde{P}(Y) > 0$ such that $\|S_{h,X}\|_{C^{1,\nu}[-Y,Y]} \le \tilde{P}\|S_{h,X}\|_{W_p^2(-Y,Y)}$. Then we obtain $\|S_{h,X}\|_{C^{1,\nu}[-Y,Y]} \le \tilde{P}\bar{P}$. By the first equation of (2.18), $\|S_{h,X}\|_{C^2[-Y,Y]} \le P$ for some positive constants $P := P(Y)$. Similar arguments prove that $\|S_{v,X}\|_{C^2[-Y,Y]}$, $\|I_{h,X}\|_{C^2[-Y,Y]}$, $\|I_{v,X}\|_{C^2[-Y,Y]} \le P$. Differentiating the first and third equations of the system (2.18) in respect of z, we get $\|S_{h,X}\|_{C^3[-Y,Y]} \le P$ and $\|S_{v,X}\|_{C^3[-Y,Y]} \le P$. According to (2.19) and (2.20), we have $\|I_{h,X}\|_{C^{2,1}[-Y,Y]} \le P$ and $\|I_{v,X}\|_{C^{2,1}[-Y,Y]} \le P$, for some $P > 0$. This finishes the proof. □

3. Existence of traveling wave solutions

To prove the existence of bounded solutions connecting E_0 and E_1^* for system (2.2), we take a sequence $\{X_n\}_{n\in\mathbb{N}^*}$ satisfying $X_n \to +\infty$ as $n \to +\infty$. From Lemma 2.8, the choice of $P(Y)$ is independent of n. Then letting $n \to +\infty$, there exists $(S_h(z), I_h(z), S_v(z), I_v(z))^T \in C^2(\mathbb{R}, \mathbb{R}^4)$ satisfying (2.2) and

$$S_h^-(z) \le S_h(z) \le S_h^0, \quad I_h^-(z) \le I_h(z) \le I_h^+(z), \tag{3.1}$$

and

$$S_v^-(z) \le S_v(z) \le S_v^0, \quad I_v^-(z) \le I_v(z) \le I_v^+(z), \tag{3.2}$$

for any $z \in \mathbb{R}$. Hence, combining (3.1) and (3.2), one gets

$$\lim_{z\to-\infty} S_h(z) = S_h^0,\ \lim_{z\to-\infty} S_v(z) = S_v^0,\ \lim_{z\to-\infty} I_h(z) = 0,\ \lim_{z\to-\infty} I_v(z) = 0.$$

Thus, (2.3) holds for $(S_h(\cdot), I_h(\cdot), S_v(\cdot), I_v(\cdot))^T$. To obtain the convergence at positive infinity, we first prove the following lemma by utilizing the approaches of [39].

Lemma 3.1. *Let $\mu := \min\{\mu_1, \mu_2, \gamma_1, \gamma_2\}$. Then the solutions of system (2.2) satisfy*

$$\frac{\Lambda}{\mu_1 + \rho_1} \le S_h(z) \le S_h^0, \quad 0 < I_h(z) \le \frac{\sqrt{D_S}\Lambda}{\sqrt{D_I}\mu}, \tag{3.3}$$

and

$$\frac{M}{\mu_2 + \rho_2} \le S_v(z) \le S_v^0, \quad 0 < I_v(z) \le \frac{\sqrt{d_S}M}{\sqrt{d_I}\mu}, \tag{3.4}$$

where $z \in \mathbb{R}$ and

$$\rho_1 := \max_{0 \le S_h(z) \le S_h^0,\, z\in\mathbb{R}} \partial_{S_h} f_1\left(S_h, \frac{\sqrt{d_S}M}{\sqrt{d_I}\mu}\right), \quad \rho_2 := \max_{0 \le S_v(z) \le S_v^0,\, z\in\mathbb{R}} \partial_{S_v} f_2\left(S_v, \frac{\sqrt{D_S}\Lambda}{\sqrt{D_I}\mu}\right).$$

Proof. Applying the strong maximum principle, we have $I_h(z)$, $I_v(z) > 0$, for any $z \in \mathbb{R}$, due to $I_h(z)$, $I_v(z) \geq 0$ and $I_h(z)$, $I_v(z) \not\equiv 0$. According to the definition of μ, then

$$\begin{cases} -D_S S_h''(z) + cS_h'(z) + \mu S_h(z) \leq \Lambda - f_1(S_h, I_v)(z), \\ -D_I I_h''(z) + cI_h'(z) + \mu I_h(z) \leq f_1(S_h, I_v)(z), \\ -d_S S_v''(z) + cS_v'(z) + \mu S_v(z) \leq M - f_2(S_v, I_h)(z), \\ -d_I I_v''(z) + cI_v'(z) + \mu I_v(z) \leq f_2(S_v, I_h)(z). \end{cases} \tag{3.5}$$

Denote $p_1(\cdot) := \Lambda - f_1(S_h, I_v)(\cdot)$ and $q_1(\cdot) := f_1(S_h, I_v)(\cdot)$. Consider the following Cauchy problems

$$\begin{cases} \partial_t w_1(t,z) - D_S \partial_z^2 w_1(t,z) + c\partial_z w_1(t,z) + \mu w_1(t,z) = p_1(z), \ \ t > 0, \ \ z \in \mathbb{R}, \\ w_1(0,z) = S_h(z), \ \ z \in \mathbb{R}, \end{cases} \tag{3.6}$$

and

$$\begin{cases} \partial_t w_2(t,z) - D_I \partial_z^2 w_2(t,z) + c\partial_z w_2(t,z) + \mu w_2(t,z) = q_1(z), \ \ t > 0, \ \ z \in \mathbb{R}, \\ w_2(0,z) = I_h(z), \ \ z \in \mathbb{R}. \end{cases} \tag{3.7}$$

Applying [9, Chapter 1, Theorems 12 and 16], one obtains

$$w_1(t,z) = \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu t}}{\sqrt{4\pi D_S t}} \mathrm{e}^{-\frac{(z-ct-\xi)^2}{4D_S t}} S_h(\xi)\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu\varpi}}{\sqrt{4\pi D_S \varpi}} \mathrm{e}^{-\frac{(z-c\varpi-\xi)^2}{4D_S\varpi}} p_1(\xi)\mathrm{d}\xi\mathrm{d}\varpi, \tag{3.8}$$

and

$$w_2(t,z) = \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu t}}{\sqrt{4\pi D_I t}} \mathrm{e}^{-\frac{(z-ct-\xi)^2}{4D_I t}} I_h(\xi)\mathrm{d}\xi + \int_0^t \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu\varpi}}{\sqrt{4\pi D_I \varpi}} \mathrm{e}^{-\frac{(z-c\varpi-\xi)^2}{4D_I\varpi}} q_1(\xi)\mathrm{d}\xi\mathrm{d}\varpi, \tag{3.9}$$

wherein $t > 0$, $z \in \mathbb{R}$. Thus, the comparison principle yields that

$$S_h(z) \leq w_1(t,z), \ \ I_h(z) \leq w_2(t,z), \ \ \forall\, t > 0, \ \ z \in \mathbb{R}.$$

Taking $t \to +\infty$ in (3.8) and (3.9), respectively, we get

$$S_h(z) \leq w_1(+\infty, z) = \frac{\Lambda}{\mu} - \int_0^{+\infty} \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu\varpi}}{\sqrt{4\pi D_S\varpi}} \mathrm{e}^{-\frac{(z-c\varpi-\xi)^2}{4D_S\varpi}} f_1(S_h, I_v)(\xi)\mathrm{d}\xi\mathrm{d}\varpi := \frac{\Lambda}{\mu} - h_{D_S}(z),$$

and

$$I_h(z) \leq w_2(+\infty, z) = \int_0^{+\infty} \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu\varpi}}{\sqrt{4\pi D_I\varpi}} \mathrm{e}^{-\frac{(z-c\varpi-\xi)^2}{4D_I\varpi}} f_1(S_h, I_v)(\xi)\mathrm{d}\xi\mathrm{d}\varpi := h_{D_I}(z),$$

for $z \in \mathbb{R}$.

By simple calculations and the assumption **(P3)**, one has

$$\begin{aligned} \sqrt{D_S} h_{D_S}(z) &= \int_0^{+\infty} \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu\varpi}}{\sqrt{4\pi\varpi}} f_1(S_h, I_v)(z - c\varpi - \xi) \mathrm{e}^{-\frac{z^2}{4D_S\varpi}} \mathrm{d}\xi\mathrm{d}\varpi \\ &\geq \int_0^{+\infty} \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mu\varpi}}{\sqrt{4\pi\varpi}} f_1(S_h, I_v)(z - c\varpi - \xi) \mathrm{e}^{-\frac{z^2}{4D_I\varpi}} \mathrm{d}\xi\mathrm{d}\varpi \\ &= \sqrt{D_I} h_{D_I}(z). \end{aligned}$$

Then

$$\sqrt{D_I} I_h(z) \le \sqrt{D_I} h_{D_I}(z) \le \sqrt{D_S} h_{D_S}(z) \le \frac{\sqrt{D_S}\Lambda}{\mu}, \quad z \in \mathbb{R},$$

i.e., $I_h(z) \le \frac{\sqrt{D_S}\Lambda}{\sqrt{D_I}\mu}$, $z \in \mathbb{R}$. The proof of inequality for I_h is finished.

Since $S_h(z)$ satisfies the inequality

$$D_S S_h''(z) - cS_h'(z) + \Lambda - (\rho_1 + \mu_1) S_h(z) \le 0, \quad z \in \mathbb{R},$$

it follows from maximum principle that $S_h(z) \ge \frac{\Lambda}{\mu_1+\rho_1}$, $z \in \mathbb{R}$. So, (3.3) holds for S_h and I_h. The inequalities for S_v and I_v can be similarly proved. This ends the proof. □

According to [22, Lemma 2.2] (or see [4, Lemma 4.7]), there is the following result.

Lemma 3.2. *Suppose $(S_h(z), I_h(z), S_v(z), I_v(z))^T$ be the solution of (2.2) satisfying (2.3). Then*

$$|I_h'(z)| \le \frac{(c+\sqrt{c^2+4\gamma_1})}{2D_I} I_h(z), \quad |I_v'(z)| \le \frac{(c+\sqrt{c^2+4\gamma_2})}{2d_I} I_v(z), \quad \text{for any } z \in \mathbb{R}.$$

Furthermore, the Harnack's inequality is established as follows

$$I_h(z) \le I_h(\bar{z}) \mathrm{e}^{\frac{(c+\sqrt{c^2+4\gamma_1})}{2D_I}|\bar{z}_1-\bar{z}_2|}, \quad I_v(z) \le I_v(\bar{z}) \mathrm{e}^{\frac{(c+\sqrt{c^2+4\gamma_2})}{2d_I}|\bar{z}_1-\bar{z}_2|},$$

for any $z, \bar{z} \in [\bar{z}_1, \bar{z}_2]$ with $\bar{z}_1 \le \bar{z}_2$, $\bar{z}_i \in \mathbb{R}$, $i = 1, 2$.

In order to prove that the solutions of (2.2) satisfy (2.4), we give the assumption as follows
(P4)

$$\left[\frac{I_h}{I_h^*} - \frac{S_h^* f_1(S_h, I_v)}{S_h f_1(S_h^*, I_v^*)}\right]\left[\frac{S_h f_1(S_h^*, I_v^*)}{S_h^* f_1(S_h, I_v)} - 1\right] \le 0 \text{ and } \left[\frac{I_v}{I_v^*} - \frac{S_v^* f_2(S_v, I_h)}{S_v f_2(S_v^*, I_h^*)}\right]\left[\frac{S_v f_2(S_v^*, I_h^*)}{S_v^* f_2(S_v, I_h)} - 1\right] \le 0.$$

Theorem 3.1. *Suppose $\mathfrak{R}_0 > 1$ and* **(P1)**–**(P4)** *hold. Then for each $c \ge c_*$, there exists a nontrivial traveling wave solution $(S_h(z), I_h(z), S_v(z), I_v(z))^T$ of system (1.2) which meets (2.3) and (2.4). Moreover,*

$$\lim_{z\to-\infty} \chi_2^{-1} \mathrm{e}^{-\zeta_1 z} I_h(z) = 1, \quad \lim_{z\to-\infty} \chi_4^{-1} \mathrm{e}^{-\zeta_1 z} I_v(z) = 1, \tag{3.10}$$

where $z = x + ct$ and c_*, ζ_1, χ_2, χ_4 are defined in Lemma 2.1.

Proof. First consider the case $c > c_*$. According to the previous discussions and Lemma 3.1, system (2.2) admits a nonnegative solution $(S_h(\cdot), I_h(\cdot), S_v(\cdot), I_v(\cdot))^T$ satisfying (2.3), (3.3) and (3.4). The strong maximum principle yields that $(S_h(z), I_h(z), S_v(z), I_v(z))^T$ is positive for all $z \in \mathbb{R}$. In addition, appealing to the following facts

$$\chi_2 \mathrm{e}^{\zeta_1 z}(1 - \mathcal{H} M_2 \mathrm{e}^{\epsilon_2 z}) \le I_h^-(z) \le I_h(z) \le I_h^+(z) \le \chi_2 \mathrm{e}^{\zeta_1 z},$$

and

$$\chi_4 \mathrm{e}^{\zeta_1 z}(1 - \mathcal{H} M_4 \mathrm{e}^{\epsilon_2 z}) \le I_v^-(z) \le I_v(z) \le I_v^+(z) \le \chi_4 \mathrm{e}^{\zeta_1 z},$$

for any $z \in \mathbb{R}$, one sees that (3.10) holds. It is thus enough to certify (2.4), i.e.,

$$S_h(z) \to S_h^*, \quad S_v(z) \to S_v^*, \quad I_h(z) \to I_h^*, \quad I_v(z) \to I_v^*, \quad \text{as } z \to +\infty.$$

For the sake of convenience, denote

$$(S_h(\cdot), I_h(\cdot), S_v(\cdot), I_v(\cdot))^T = (U_1(\cdot), U_2(\cdot), U_3(\cdot), U_4(\cdot))^T := U(\cdot).$$

To apply the LaSalle's invariance principle, letting $U_i'(\cdot) := V_i(\cdot)$, $i = 1, 2, 3, 4$. Hence, system (2.2) is transformed into

$$\begin{cases} U_1'(z) = V_1(z), \\ D_S V_1'(z) = cV_1(z) - \Lambda + f_1(U_1, U_4)(z) + \mu_1 U_1(z), \\ U_2'(z) = V_2(z), \\ D_I V_2'(z) = cV_2(z) - f_1(U_1, U_4)(z) + \gamma_1 U_2(z), \\ U_3'(z) = V_3(z), \\ d_S V_3'(z) = cV_3(z) - M + f_2(U_3, U_2)(z) + \mu_2 U_3(z), \\ U_4'(z) = V_4(z), \\ d_I V_4'(z) = cV_4(z) - f_2(U_3, U_2)(z) + \gamma_2 U_4(z). \end{cases}$$

Define a Lyapunov functional $\mathcal{L}(z)$ as follows

$$\mathcal{L}(z) := \mathcal{L}_1(z) + \mathcal{L}_2(z) + \mathcal{L}_3(z) + \mathcal{L}_4(z), \ \ z \in \mathbb{R},$$

where

$$\begin{aligned} \mathcal{L}_1(z) &:= cU_1(z) - D_S V_1(z) + \frac{S_h^* D_S V_1(z)}{U_1(z)} - cS_h^* \ln U_1(z), \\ \mathcal{L}_2(z) &:= cU_2(z) - D_I V_2(z) + \frac{I_h^* D_I V_2(z)}{U_2(z)} - cI_h^* \ln U_2(z), \\ \mathcal{L}_3(z) &:= cU_3(z) - d_S V_3(z) + \frac{S_v^* d_S V_3(z)}{U_3(z)} - cS_v^* \ln U_3(z), \end{aligned}$$

and

$$\mathcal{L}_4(z) := cU_4(z) - d_I V_4(z) + \frac{I_v^* d_I V_4(z)}{U_4(z)} - cI_v^* \ln U_4(z).$$

Note that U_i is bounded in $C^2(\mathbb{R})$, $i = 1, 2, 3, 4$. Since the function $x - 1 - \ln x$ is nonnegative for all $x > 0$, one gets

$$\begin{aligned} \mathcal{L}_1(z) &\geq cU_1(z) - D_S \|V_1\|_{L^\infty} - \frac{S_h^* D_S \|V_1\|_{L^\infty}}{U_1(z)} - cS_h^* \ln U_1(z) \\ &= cS_h^* \left[\frac{U_1(z)}{S_h^*} - \ln U_1(z) \right] - D_S \|V_1\|_{L^\infty} - \frac{S_h^* D_S \|V_1\|_{L^\infty}}{U_1(z)} \\ &= cS_h^* \left[\frac{U_1(z)}{S_h^*} - 1 - \ln \frac{U_1(z)}{S_h^*} - \ln S_h^* + 1 \right] - D_S \|V_1\|_{L^\infty} - \frac{S_h^* D_S \|V_1\|_{L^\infty}}{U_1(z)} \\ &\geq cS_h^*(1 - \ln S_h^*) - \frac{S_h^* D_S (\mu_1 + \rho_1) \|V_1\|_{L^\infty}}{\Lambda} - D_S \|V_1\|_{L^\infty}, \ \ z \in \mathbb{R}, \end{aligned}$$

which is due to Lemma 3.1. In addition,

$$\begin{aligned} \mathcal{L}_2(z) &\geq cU_2(z) - D_I \|V_2\|_{L^\infty} - \frac{I_h^* D_I |V_2(z)|}{U_2(z)} - cI_h^* \ln U_2(z) \\ &= cI_h^* \left[\frac{U_2(z)}{I_h^*} - \ln U_2(z) \right] - D_I \|V_2\|_{L^\infty} - \frac{I_h^* D_I |V_2(z)|}{U_2(z)} \\ &\geq cI_h^*(1 - \ln I_h^*) - \frac{(c + \sqrt{c^2 + 4\gamma_1}) I_h^*}{2} - D_I \|V_2\|_{L^\infty}, \ \ z \in \mathbb{R}, \end{aligned}$$

which is owing to Lemma 3.2. Similarly, we can show that $\mathcal{L}_3(z)$ and $\mathcal{L}_4(z)$ have lower bounds. Then $\mathcal{L}(z)$ has a lower bound. Because

$$\Lambda = f_1(S_h^*, I_v^*) + \mu_1 S_h^*,\ \gamma_1 I_h^* = f_1(S_h^*, I_v^*),\ M = f_2(S_v^*, I_h^*) + \mu_2 S_v^*,\ \gamma_2 I_v^* = f_2(S_v^*, I_h^*),$$

after elementary but tedious computations, we obtain

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{L}(z)}{\mathrm{d}z} &= \frac{\mathrm{d}\mathcal{L}_1(z)}{\mathrm{d}z} + \frac{\mathrm{d}\mathcal{L}_2(z)}{\mathrm{d}z} + \frac{\mathrm{d}\mathcal{L}_3(z)}{\mathrm{d}z} + \frac{\mathrm{d}\mathcal{L}_4(z)}{\mathrm{d}z} \\
&= -\mu_1 \frac{[S_h^* - U_1(z)]^2}{U_1(z)} - \frac{S_h^* D_S V_1^2(z)}{U_1^2(z)} - \frac{I_h^* D_I V_2^2(z)}{U_2^2(z)} \\
&\quad + f_1(S_h^*, I_v^*)\left[3 - \frac{S_h^*}{U_1(z)} - \frac{I_h^* f_1(U_1, U_4)(z)}{U_2(z) f_1(S_h^*, I_v^*)} - \frac{U_1(z) f_1(S_h^*, I_v^*) U_2(z)}{S_h^* f_1(U_1, U_4)(z) I_h^*}\right] \\
&\quad + f_1(S_h^*, I_v^*)\left[\frac{U_1(z) f_1(S_h^*, I_v^*) U_2(z)}{S_h^* f_1(U_1, U_4)(z) I_h^*} - \frac{U_2(z)}{I_h^*} - 1 + \frac{S_h^* f_1(U_1, U_4)(z)}{U_1(z) f_1(S_h^*, I_v^*)}\right] \\
&\quad - \mu_2 \frac{[S_v^* - U_3(z)]^2}{U_3(z)} - \frac{S_v^* d_S V_3^2(z)}{U_3^2(z)} - \frac{I_v^* d_I V_4^2(z)}{U_4^2(z)} \\
&\quad + f_2(S_v^*, I_h^*)\left[3 - \frac{S_v^*}{U_3(z)} - \frac{I_v^* f_2(U_3, U_2)(z)}{U_4(z) f_2(S_v^*, I_h^*)} - \frac{U_3(z) f_2(S_v^*, I_h^*) U_4(z)}{S_v^* f_2(U_3, U_2)(z) I_v^*}\right] \\
&\quad + f_2(S_v^*, I_h^*)\left[\frac{U_3(z) f_2(S_v^*, I_h^*) U_4(z)}{S_v^* f_2(U_3, U_2)(z) I_v^*} - \frac{U_4(z)}{I_v^*} - 1 + \frac{S_v^* f_2(U_3, U_2)(z)}{U_3(z) f_2(S_v^*, I_h^*)}\right],
\end{aligned}$$

where

$$\begin{aligned}
&\frac{U_1(z) f_1(S_h^*, I_v^*) U_2(z)}{S_h^* f_1(U_1, U_4)(z) I_h^*} - \frac{U_2(z)}{I_h^*} - 1 + \frac{S_h^* f_1(U_1, U_4)(z)}{U_1(z) f_1(S_h^*, I_v^*)} \\
&\quad = \left[\frac{U_1(z) f_1(S_h^*, I_v^*)}{S_h^* f_1(U_1, U_4)(z)} - 1\right]\left[\frac{U_2(z)}{I_h^*} - \frac{S_h^* f_1(U_1, U_4)(z)}{U_1(z) f_1(S_h^*, I_v^*)}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{U_3(z) f_2(S_v^*, I_h^*) U_4(z)}{S_v^* f_2(U_3, U_2)(z) I_v^*} - \frac{U_4(z)}{I_v^*} - 1 + \frac{S_v^* f_2(U_3, U_2)(z)}{U_3(z) f_2(S_v^*, I_h^*)} \\
&\quad = \left[\frac{U_3(z) f_2(S_v^*, I_h^*)}{S_v^* f_2(U_3, U_2)(z)} - 1\right]\left[\frac{U_4(z)}{I_v^*} - \frac{S_v^* f_2(U_3, U_2)(z)}{U_3(z) f_2(S_v^*, I_h^*)}\right].
\end{aligned}$$

With the help of the mean inequality and the assumption **(P4)**, we know that $\mathrm{d}\mathcal{L}(z)/\mathrm{d}z \le 0$ and $\mathrm{d}\mathcal{L}(z)/\mathrm{d}z \equiv 0$ iff $U_1(z) \equiv S_h^*$, $U_2(z) \equiv I_h^*$, $U_3(z) \equiv S_v^*$ and $U_4(z) \equiv I_v^*$. Therefore, the largest compact invariant set

$$\Theta_E = \left\{U(z)|\frac{\mathrm{d}\mathcal{L}(z)}{\mathrm{d}z} = 0,\ z \in \mathbb{R}\right\} \equiv \{E_1^*\} \equiv \{(S_h^*, I_h^*, S_v^*, I_v^*)^T\}.$$

Then the LaSalle's invariance principle implies that

$$(U_1(+\infty), U_2(+\infty), U_3(+\infty), U_4(+\infty))^T = (S_h^*, I_h^*, S_v^*, I_v^*)^T.$$

For the case $c = c_*$. Similar to the arguments of [39, Theorem 2.14], we can obtain the existence of TWS in this case. This completes the proof. □

4. Nonexistence of traveling wave solutions

4.1. Nonexistence when $\mathfrak{R}_0 \le 1$ and $c > 0$

The main results of this subsection are as follows:

Theorem 4.1. *Assume that $\mathfrak{R}_0 \le 1$. Then for any $c > 0$, system (1.2) has no traveling wave solutions with speed c which meet (2.3) and (2.4).*

Proof. On the contrary, we suppose that there is a $(S_h(z), I_h(z), S_v(z), I_v(z))^T$ for (1.2) satisfying (2.3) and (2.4), $z = x + ct$. Denote

$$\bar{I}_h := \sup_{z\in\mathbb{R}} I_h(z), \;\; \bar{I}_v := \sup_{z\in\mathbb{R}} I_v(z).$$

For $\mathfrak{R}_0 < 1$. By **(P1)–(P2)**, it follows from the second and fourth equations of (2.2) that

$$\begin{cases} cI_h'(z) - D_I I_h''(z) + \gamma_1 I_h(z) - k_1 \bar{I}_v \le 0, \\ cI_v'(z) - d_I I_v''(z) + \gamma_2 I_v(z) - k_2 \bar{I}_h \le 0, \end{cases} \tag{4.1}$$

wherein $z \in \mathbb{R}$, $k_1 = \partial_{I_v} f_1(S_h^0, 0)$ and $k_2 = \partial_{I_h} f_2(S_v^0, 0)$. Utilizing the comparison principle, one obtains

$$I_h(z) \le \frac{k_1 \bar{I}_v}{\gamma_1}, \;\; I_v(z) \le \frac{k_2 \bar{I}_h}{\gamma_2}, \;\; z \in \mathbb{R}.$$

That is, $(I_h(z), I_v(z))^T \le \mathcal{M}(\bar{I}_h, \bar{I}_v)^T$ for any $z \in \mathbb{R}$, where $\mathcal{M}$ is defined in Sect. 2, which leads to $(\bar{I}_h, \bar{I}_v)^T \le \mathcal{M}^n(\bar{I}_h, \bar{I}_v)^T$, $n \in \mathbb{N}^*$. Since $\mathfrak{R}_0$ is the spectral radius of $\mathcal{M}$ and the matrix $\mathcal{M}$ is nonnegative and irreducible, the Perron–Frobenius theorem implies that there is a positive eigenvector $\mathcal{U} = (u_1, u_2)^T$, corresponding to $\mathfrak{R}_0$ such that $\mathcal{M}\mathcal{U} = \mathfrak{R}_0\mathcal{U}$. Moreover, by Lemma 3.1, there exists a constant $C_3 > 0$ such that $(\bar{I}_h, \bar{I}_v)^T \le C_3\mathcal{U}$. Hence, we have

$$(\bar{I}_h, \bar{I}_v)^T \le \mathcal{M}^n(\bar{I}_h, \bar{I}_v)^T \le C_3\mathcal{M}^n\mathcal{U} = C_3\mathfrak{R}_0^n\mathcal{U} \to 0, \;\text{ as } n \to +\infty,$$

which is owing to $\mathfrak{R}_0 < 1$. So, one gets $\bar{I}_h = \bar{I}_v = 0$. If not, then $\bar{I}_h$ or $\bar{I}_v$ is a positive constant. Without loss of generality, assume $\bar{I}_h > 0$. From the fact $\mathfrak{R}_0 < 1$, there is a sufficiently large $n_0 \in \mathbb{N}^*$ such that $\mathfrak{R}_0^{n_0} < \bar{I}_h/2C_3u_1$. Thus, we get $\bar{I}_h \le C_3\mathfrak{R}_0^{n_0}u_1 < C_3u_1 \cdot (\bar{I}_h/2C_3u_1) = \bar{I}_h/2$ which is a contradiction. Thus, we obtain $\bar{I}_h = 0$. Similarly, $\bar{I}_v = 0$. This contradicts to the fact $I_h(z)$, $I_v(z) > 0$ for all $z \in \mathbb{R}$.

For $\mathfrak{R}_0 = 1$. Likely above discussions, there is an eigenvector $\mathcal{V} = (v_1, v_2)^T$ with $v_1 > 0$ and $v_2 > 0$ such that $\mathcal{M}\mathcal{V} = \mathcal{V}$. Direct calculations show that

$$\gamma_1 = \frac{k_1 v_2}{v_1}, \;\; \gamma_2 = \frac{k_2 v_1}{v_2}. \tag{4.2}$$

Choosing a sequence $\{z_n\} \subset \mathbb{R}$, $n \in \mathbb{N}^*$ such that

$$\lim_{n\to+\infty} I_h(z_n) = \bar{I}_h = \sup_{z\in\mathbb{R}} I_h(z).$$

To derive a contradiction, one intends to prove $\bar{I}_h = 0$. Arguing by contradiction, assuming $\bar{I}_h > 0$. Consider the following function sequence

$$(S_{h,n}(\cdot), I_{h,n}(\cdot), S_{v,n}(\cdot), I_{v,n}(\cdot))^T := (S_h(\cdot + z_n), I_h(\cdot + z_n), S_v(\cdot + z_n).I_v(\cdot + z_n))^T,$$

By using the boundedness of $(S_h, I_h, S_v, I_v)^T$ (see Lemma 3.1) and elliptic estimates, there exists a subsequence $(S_{h,n_j}, I_{h,n_j}, S_{v,n_j}, I_{v,n_j})^T$, $j \in \mathbb{N}^*$ of $(S_{h,n}, I_{h,n}, S_{v,n}, I_{v,n})^T$ and $(\tilde{S}_h, \tilde{I}_h, \tilde{S}_v, \tilde{I}_v)^T$ such that

$$\lim_{j\to+\infty} (S_{h,n_j}(z), I_{h,n_j}(z), S_{v,n_j}(z), I_{v,n_j}(z))^T = (\tilde{S}_h(z), \tilde{I}_h(z), \tilde{S}_v(z), \tilde{I}_v(z))^T$$

in $C^2_{\mathrm{loc}}(\mathbb{R})$ and $(\tilde{S}_h, \tilde{I}_h, \tilde{S}_v, \tilde{I}_v)^T$ satisfies

$$\begin{cases} c\tilde{S}_h'(z) = D_S\tilde{S}_h''(z) + \Lambda - f_1(\tilde{S}_h, \tilde{I}_v)(z) - \mu_1\tilde{S}_h(z), \\ c\tilde{I}_h'(z) = D_I\tilde{I}_h''(z) + f_1(\tilde{S}_h, \tilde{I}_v)(z) - \gamma_1\tilde{I}_h(z), \\ c\tilde{S}_v'(z) = d_S\tilde{S}_v''(z) + M - f_2(\tilde{S}_v, \tilde{I}_h)(z) - \mu_2\tilde{S}_v(z), \\ c\tilde{I}_v'(z) = d_I\tilde{I}_v''(z) + f_2(\tilde{S}_v, \tilde{I}_h)(z) - \gamma_2\tilde{I}_v(z), \end{cases} \tag{4.3}$$

where $\tilde{I}_h(0) = \bar{I}_h$, $\tilde{I}_h(z) \le \bar{I}_h$, $0 < \tilde{S}_h(z) \le S_h^0$ and $0 < \tilde{S}_v(z) \le S_v^0$, for $z \in \mathbb{R}$. Applying the comparison principle to the second equation of (4.1) and using (4.2) yield that $k_2\bar{I}_h \ge \gamma_2\bar{I}_v = k_2\bar{I}_v v_1/v_2$, then $\bar{I}_v \le \bar{I}_h v_2/v_1$. Since

$$\begin{aligned} 0 &= D_I\tilde{I}_h''(z) - c\tilde{I}_h'(z) + f_1(\tilde{S}_h, \tilde{I}_v)(z) - \gamma_1\tilde{I}_h(z) \\ &\le D_I\tilde{I}_h''(z) - c\tilde{I}_h'(z) + \partial_{I_v} f_1(\tilde{S}_h, 0)(z)\tilde{I}_v(z) - \gamma_1\tilde{I}_h(z), \end{aligned}$$

it follows from (4.2) that

$$\begin{aligned} 0 &\le D_I\tilde{I}_h''(0) - c\tilde{I}_h'(0) + \partial_{I_v} f_1(\tilde{S}_h, 0)(0)\tilde{I}_v(0) - \gamma_1\tilde{I}_h(0) \\ &\le D_I\tilde{I}_h''(0) + \frac{\partial_{I_v} f_1(\tilde{S}_h, 0)(0)v_2}{v_1}\bar{I}_h - \frac{\partial_{I_v} f_1(S_h^0, 0)v_2}{v_1}\bar{I}_h \\ &= D_I\tilde{I}_h''(0) + \left[\partial_{I_v} f_1(\tilde{S}_h, 0)(0) - \partial_{I_v} f_1(S_h^0, 0)\right]\frac{v_2}{v_1}\bar{I}_h \end{aligned}$$

Then $\tilde{S}_h(0) \ge S_h^0$ due to $\tilde{I}_h''(0) \le 0$ and **(P1)**. It is impossible that $\tilde{S}_h(0) > S_h^0$ as $\tilde{S}_h(z) \le S_h^0$, $z \in \mathbb{R}$. Thus, we get $\tilde{S}_h(0) = S_h^0$. The strong maximum principle implies $\tilde{S}_h(z) \equiv S_h^0$, $z \in \mathbb{R}$. Substituting it into the first equation of (4.3) and combining **(P2)**, we obtain $\tilde{I}_v(z) \equiv 0$, $z \in \mathbb{R}$. Then $\tilde{I}_h(z) \equiv 0$, $z \in \mathbb{R}$ from the fourth equation of (4.3) and **(P2)**. Therefore, $\bar{I}_h = 0$ which contradicts the assumption $\bar{I}_h > 0$. This finishes the proof. □

4.2. Nonexistence when $\mathfrak{R}_0 > 1$ and $0 < c < c_*$

To investigate the nonexistence, we first to show the following lemma by the methods of [4].

Lemma 4.1. *Suppose $(S_h(z), I_h(z), S_v(z), I_v(z))^T$ be the solution of system (2.2). Then there exists a constant $B \gg 1$ such that*

$$\frac{1}{B}I_h(z) \le I_v(z) \le BI_h(z), \text{ for any } z \in \mathbb{R}.$$

Proof. By (2.2), one has

$$\begin{cases} D_I I_h''(z) - cI_h'(z) - \gamma_1 I_h(z) + f_1(S_h, I_v)(z) = 0, & z \in \mathbb{R}, \\ d_I I_v''(z) - cI_v'(z) - \gamma_2 I_v(z) + f_2(S_v, I_h)(z) = 0, & z \in \mathbb{R}. \end{cases}$$

According to the Harnack's inequality in Lemma 3.2, there is a $K > 0$ such that

$$I_i(\xi) \ge KI_i(z), \ \xi \in [z-1, z+1], \ z \in \mathbb{R}, \ i = h, v.$$

The method of constant variation yields that

$$I_h(z) = \frac{1}{\Pi_1}\left[\int_{-\infty}^{z} e^{\zeta_1^-(z-\xi)} f_1(S_h, I_v)(\xi)d\xi + \int_{z}^{+\infty} e^{\zeta_1^+(z-\xi)} f_1(S_h, I_v)(\xi)d\xi\right],$$

and

$$I_v(z) = \frac{1}{\Pi_2}\left[\int\limits_{-\infty}^{z} e^{\zeta_2^-(z-\xi)} f_2(S_v, I_h)(\xi)d\xi + \int\limits_{z}^{+\infty} e^{\zeta_2^+(z-\xi)} f_2(S_v, I_h)(\xi)d\xi\right],$$

wherein

$$\Pi_1 = D_I(\zeta_1^+ - \zeta_1^-), \quad \Pi_2 = d_I(\zeta_2^+ - \zeta_2^-),$$

and

$$\zeta_1^\pm = \frac{c \pm \sqrt{c^2 + 4D_I\gamma_1}}{2D_I}, \quad \zeta_2^\pm = \frac{c \pm \sqrt{c^2 + 4d_I\gamma_2}}{2d_I}.$$

Applying Lemma 3.1 and **(P1)–(P2)**, we have

$$\begin{aligned} I_h(z) &= \frac{1}{\Pi_1}\left[\int\limits_{-\infty}^{z} e^{\zeta_1^-(z-\xi)} f_1(S_h, I_v)(\xi)d\xi + \int\limits_{z}^{+\infty} e^{\zeta_1^+(z-\xi)} f_1(S_h, I_v)(\xi)d\xi\right] \\ &\geq \frac{1}{\Pi_1}\left[\int\limits_{-\infty}^{z} e^{\zeta_1^-(z-\xi)} \hat{\rho}_1 I_v(\xi)d\xi + \int\limits_{z}^{+\infty} e^{\zeta_1^+(z-\xi)} \hat{\rho}_1 I_v(\xi)d\xi\right] \\ &\geq \frac{1}{\Pi_1}\left[\int\limits_{z-1}^{z} e^{\zeta_1^-(z-\xi)} \hat{\rho}_1 I_v(\xi)d\xi + \int\limits_{z}^{z+1} e^{\zeta_1^+(z-\xi)} \hat{\rho}_1 I_v(\xi)d\xi\right] \\ &\geq \frac{\hat{\rho}_1 K I_v(z)}{\Pi_1}\left[\int\limits_{z-1}^{z} e^{\zeta_1^-(z-\xi)} d\xi + \int_{z}^{z+1} e^{\zeta_1^+(z-\xi)} d\xi\right] \\ &= \frac{\hat{\rho}_1 K}{\Pi_1}\left[\int_0^1 e^{\zeta_1^-\xi} d\xi + \int_{-1}^{0} e^{\zeta_1^+\xi} d\xi\right] I_v(z) := A_1 I_v(z), \end{aligned} \tag{4.4}$$

and similarly

$$I_v(z) \geq \frac{\hat{\rho}_2 K}{\Pi_2}\left[\int_0^1 e^{\zeta_2^-\xi} d\xi + \int\limits_{-1}^{0} e^{\zeta_2^+\xi} d\xi\right] I_h(z) := A_2 I_h(z), \tag{4.5}$$

for all $z \in \mathbb{R}$ with

$$\hat{\rho}_1 = \partial_{I_v} f_1\left(\frac{\Lambda}{\mu_1 + \rho_1}, \frac{\sqrt{d_S}M}{\sqrt{d_I}\mu}\right), \quad \hat{\rho}_2 = \partial_{I_h} f_2\left(\frac{M}{\mu_2 + \rho_2}, \frac{\sqrt{D_S}\Lambda}{\sqrt{D_I}\mu}\right).$$

Choosing B satisfying $B \gg \max\{A_1^{-1}, A_2^{-1}, 1\}$, then combining (4.4) and (4.5) implies that the conclusion is valid. This ends the proof. □

The main results of this subsection are as follows:

Theorem 4.2. *Assume that $\Re_0 > 1$. Then system (1.2) has no nontrivial traveling wave solutions with speed $c < c_*$ connecting E_0 and E_1^*.*

Proof. Suppose by way of contradiction that system (1.2) has a traveling wave solution $(S_h(z), I_h(z), S_v(z), I_v(z))^T$ with speed c connecting E_0 and E_1^*, $z = x + ct$.

Consider the following sequence

$$(I_{h,m}(z), I_{v,m}(z))^T := \left(\frac{I_h(z-m)}{I_h(-m)}, \frac{I_v(z-m)}{I_h(-m)}\right)^T, \quad z \in \mathbb{R}, \quad m \in \mathbb{N}^*.$$

Since $(S_h(\cdot - m), I_h(\cdot - m), S_v(\cdot - m), I_v(\cdot - m))^T$ is also a solution of (2.2) for any $m \in \mathbb{N}^*$, and

$$f_1(S_h(z-m), I_v(z-m)) = \partial_{I_v} f_1(S_h(z-m), \xi_{I_v}) I_v(z-m),$$

and

$$f_2(S_v(z-m), I_h(z-m)) = \partial_{I_h} f_2(S_v(z-m), \xi_{I_h}) I_h(z-m),$$

where $0 \le \xi_{I_v} \le I_v(z-m)$, $0 \le \xi_{I_h} \le I_h(z-m)$, one obtains that $(I_{h,m}(z), I_{v,m}(z))^T$ satisfies

$$\begin{cases} D_I I''_{h,m}(z) - cI'_{h,m}(z) - \gamma_1 I_{h,m}(z) + \partial_{I_v} f_1(S_h(z-m), \xi_{I_v}) I_{v,m}(z) = 0, & z \in \mathbb{R}, \\ d_I I''_{v,m}(z) - cI'_{v,m}(z) - \gamma_2 I_{v,m}(z) + \partial_{I_h} f_2(S_v(z-m), \xi_{I_h}) I_{h,m}(z) = 0, & z \in \mathbb{R}. \end{cases}$$

On the one hand, Lemma 3.2 indicates that there is a constant $C_4 > 0$ such that

$$I_{h,m}(z) = \frac{I_h(z-m)}{I_h(-m)} \le \frac{1}{I_h(-m)} C_4 I_h(-m) \mathrm{e}^{C_4|z-m-(-m)|} = C_4 \mathrm{e}^{C_4|z|},$$

and then $|I'_{h,m}(z)| \le C_4 I_{h,m}(z) \le C_4^2 \mathrm{e}^{C_4|z|}$. Moreover, from Lemmas 3.2 and 4.1, there exists a $C_5 > 0$ such that

$$I_{v,m}(z) = \frac{I_v(z-m)}{I_h(-m)} \le \frac{1}{I_h(-m)} C_5 I_v(-m) \mathrm{e}^{C_5|z-m-(-m)|} = \frac{I_v(-m)}{I_h(-m)} C_5 \mathrm{e}^{C_5|z|} \le BC_5 \mathrm{e}^{C_5|z|}.$$

So, $|I'_{v,m}(z)| \le C_5 I_{v,m}(z) \le BC_5^2 \mathrm{e}^{C_4|z|}$. Accordingly, one has

$$\max\{I_{h,m}(z), I_{v,m}(z), |I'_{h,m}(z)|, |I'_{v,m}(z)|\} \le C_6 \mathrm{e}^{C_6|z|}, \text{ for some } C_6 > 0.$$

Therefore, applying the standard elliptic estimates, there exists a $(I_{h,*}(\cdot), I_{v,*}(\cdot))^T$ such that $(I_{h,m}(\cdot), I_{v,m}(\cdot))^T \to (I_{h,*}(\cdot), I_{v,*}(\cdot))^T$ as $m \to +\infty$ in $C^2_{\text{loc}}(\mathbb{R}^2)$. Owing to $S_h(\cdot - m) \to S_h^0$, $S_v(\cdot - m) \to S_v^0$ and ξ_{I_h}, $\xi_{I_v} \to 0$, $m \to +\infty$, $(I_{h,*}(\cdot), I_{v,*}(\cdot))^T$ satisfies

$$\begin{cases} D_I I''_{h,*}(z) - cI'_{h,*}(z) - \gamma_1 I_{h,*}(z) + k_1 I_{v,*}(z) = 0, & z \in \mathbb{R}, \\ d_I I''_{v,*}(z) - cI'_{v,*}(z) - \gamma_2 I_{v,*}(z) + k_2 I_{h,*}(z) = 0, & z \in \mathbb{R}, \\ I_{h,*}(z) > 0, \ \ I_{v,*}(z) > 0, & z \in \mathbb{R}, \\ I_{h,*}(0) = 1, \ \ \dfrac{1}{B} \le \dfrac{I_{h,*}(z)}{I_{v,*}(z)} \le B, & z \in \mathbb{R}. \end{cases} \tag{4.6}$$

where $k_1 = \partial_{I_v} f_1(S_h^0, 0)$ and $k_2 = \partial_{I_h} f_2(S_v^0, 0)$, B is determined in Lemma 4.1. To complete the proof, we will prove that either $I_{h,*}(z)$ or $I_{v,*}(z)$ changes sign for some $z \in \mathbb{R}$.

By (4.6), it is not difficult to see that

$$\frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} I_{h,*} \\ I_{v,*} \\ I'_{h,*} \\ I'_{v,*} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\gamma_1}{D_I} & -\frac{k_1}{D_I} & \frac{c}{D_I} & 0 \\ -\frac{k_2}{d_I} & \frac{\gamma_2}{d_I} & 0 & \frac{c}{d_I} \end{pmatrix} \begin{pmatrix} I_{h,*} \\ I_{v,*} \\ I'_{h,*} \\ I'_{v,*} \end{pmatrix} := \mathbb{A} \begin{pmatrix} I_{h,*} \\ I_{v,*} \\ I'_{h,*} \\ I'_{v,*} \end{pmatrix}.$$

Thus, the characteristic equation for matrix $\mathbb{A}$ is

$$Q(\zeta) = \zeta^2 \left(\zeta - \frac{c}{D_I}\right)\left(\zeta - \frac{c}{d_I}\right) - \zeta\left(\zeta - \frac{c}{D_I}\right)\frac{\gamma_2}{d_I} - \zeta\left(\zeta - \frac{c}{d_I}\right)\frac{\gamma_1}{D_I} - \frac{\bar{k} - \bar{\gamma}}{D_I d_I},$$

where $\bar{k} = k_1 k_2$, $\bar{\gamma} = \gamma_1 \gamma_2$. Since $\mathfrak{R}_0 = \sqrt{\bar{k}/\bar{\gamma}} > 1$, $Q(0) = \frac{\bar{\gamma} - \bar{k}}{D_I d_I} < 0$. Because $Q(\zeta) \to +\infty$ as $\zeta \to \pm\infty$, the matrix $\mathbb{A}$ has at least two real eigenvalues with opposite signs. Moreover, if $\zeta \in \mathbb{C}$ (the set of complex number) is an eigenvalue of matrix $\mathbb{A}$ and the corresponding eigenvector is denoted by $(\hat{\chi}_2, \hat{\chi}_4, \tilde{\chi}_2, \tilde{\chi}_4)^T$, then we have $(\tilde{\chi}_2, \tilde{\chi}_4)^T = (\zeta\hat{\chi}_2, \zeta\hat{\chi}_4)^T$ and

$$\mathcal{A}(\zeta) \begin{pmatrix} \hat{\chi}_2 \\ \hat{\chi}_4 \end{pmatrix} := \begin{pmatrix} D_I \zeta^2 - \gamma_1 & k_1 \\ k_2 & d_I \zeta^2 - \gamma_2 \end{pmatrix} \begin{pmatrix} \hat{\chi}_2 \\ \hat{\chi}_4 \end{pmatrix} = c\zeta \begin{pmatrix} \hat{\chi}_2 \\ \hat{\chi}_4 \end{pmatrix}.$$

Thus, $c\zeta$ is the eigenvalue of matrix $\mathcal{A}(\zeta)$. Observe that, for every $\zeta \in \mathbb{R}$, the two real eigenvalues of matrix $\mathcal{A}(\zeta)$ are

$$\alpha_-(\zeta) = \frac{1}{2}\left\{[(D_I + d_I)\zeta^2 - (\gamma_1+\gamma_2)] - \sqrt{[(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)]^2 + 4k_1k_2}\right\},$$

and

$$\alpha_+(\zeta) = \frac{1}{2}\left\{[(D_I + d_I)\zeta^2 - (\gamma_1+\gamma_2)] + \sqrt{[(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)]^2 + 4k_1k_2}\right\}.$$

In particular, if $\zeta \in \mathbb{R}$, then $c\zeta \in \{\alpha_-(\zeta), \alpha_+(\zeta)\}$. Following from the fact $\alpha_-(\zeta) < \alpha_+(\zeta)$, $\zeta \in \mathbb{R}$ gives that the dimension of the eigenspace formed by any real eigenvalue ζ of matrix $\mathbb{A}$ is always one, i.e., $\mathrm{Dim}(\mathbb{G}_\zeta) = 1$.

Next to show $c\zeta = \alpha_-(\zeta)$. If not, then $c\zeta = \alpha_+(\zeta)$. We claim that $\alpha_+(\zeta) > 0$, for all $\zeta \in \mathbb{R}$. Indeed, simple calculations yield that

$$\begin{aligned}
\zeta\alpha_+'(\zeta) &= \zeta^2\left\{(D_I + d_I) + \frac{[(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)](D_I - d_I)}{\sqrt{[(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)]^2 + 4k_1k_2}}\right\} \\
&> \zeta^2\frac{(D_I + d_I)|(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)| + [(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)](D_I - d_I)}{\sqrt{[(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)]^2 + 4k_1k_2}} \\
&\geq \zeta^2\frac{2d_I|(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)|}{\sqrt{[(D_I - d_I)\zeta^2 - (\gamma_1-\gamma_2)]^2 + 4k_1k_2}}
\end{aligned}$$

So, $\zeta\alpha_+'(\zeta) > 0$ for any $\zeta \in \mathbb{R}\backslash\{0\}$. Therefore, we obtain

$$\begin{aligned}
\alpha_+(\zeta) &> \alpha_+(0) \\
&= \frac{1}{2}\left[-(\gamma_1+\gamma_2) + \sqrt{(\gamma_1-\gamma_2)^2 + 4k_1k_2}\right] \\
&> \frac{1}{2}\left[-(\gamma_1+\gamma_2) + \sqrt{(\gamma_1-\gamma_2)^2 + 4\gamma_1\gamma_2}\right] \\
&= 0, \ \ \zeta \in (0, +\infty),
\end{aligned}$$

which is due to $\mathfrak{R}_0 > 1$. Then $\alpha_+(\zeta) > 0$ since $\alpha_+(\zeta)$ is an even function of ζ. So, $c = \alpha_+(\zeta)/\zeta$. Similar to the arguments of [4, Lemma 2.2], one can obtain that $c = \alpha_+(\zeta)/\zeta \geq c_*$ which contradicts with $c < c_*$. In conclusion, we get $c\zeta = \alpha_-(\zeta)$.

Since $(\hat{\chi}_2, \hat{\chi}_4)^T = (1, \beta_\zeta)^T$ and $(\tilde{\chi}_2, \tilde{\chi}_4)^T = (\zeta\hat{\chi}_2, \zeta\hat{\chi}_4)^T$, one has

$$\mathbb{G}_\zeta = \mathit{span}\{(1, \beta_\zeta, \zeta, \zeta\beta_\zeta)^T\},$$

where $\beta_\zeta = [\alpha_-(\zeta) - (D_I\zeta^2 - \gamma_1)]/k_1$. Using the following facts

$$[\alpha - (D_I\zeta^2 - \gamma_1)][\alpha - (d_I\zeta^2 - \gamma_2)] = k_1k_2 > 0, \ \ \text{for } \alpha \in \{\alpha_-(\zeta), \alpha_+(\zeta)\},$$

and

$$\alpha_-(\zeta) + \alpha_+(\zeta) = (D_I\zeta^2 - \gamma_1) + (d_I\zeta^2 - \gamma_2),$$

it is not difficult to verify that

$$\alpha_-(\zeta) < \min\{D_I\zeta^2 - \gamma_1, d_I\zeta^2 - \gamma_2\} \leq \max\{D_I\zeta^2 - \gamma_1, d_I\zeta^2 - \gamma_2\} < \alpha_+(\zeta).$$

Then $\beta_\zeta = [\alpha_-(\zeta) - (D_I\zeta^2 - \gamma_1)]/k_1 < 0$. For convenience, set $G_\zeta := (1, \beta_\zeta)^T$.

To show that either $I_{h,*}(z)$ or $I_{v,*}(z)$ changes sign, based on the distribution of eigenvalues of matrix $\mathbb{A}$, we prove it in two cases.

Case 1 Matrix $\mathbb{A}$ has a pair of complex eigenvalues.

From the previous discussions, we know that $\mathbb{A}$ has a pair of positive and negative eigenvalues, which can be denoted as $\zeta_- < 0 < \zeta_+$. Assuming that $\zeta = \omega \pm i\theta$ with $\theta > 0$ are the two complex eigenvalues, then $(I_{h,*}(z), I_{v,*}(z))^T$ can be expressed as

$$\begin{pmatrix} I_{h,*}(z) \\ I_{v,*}(z) \end{pmatrix} = a_1 \mathrm{e}^{\zeta_- z} G_{\zeta_-} + a_2 \mathrm{e}^{\zeta_+ z} G_{\zeta_+} + a_3 \mathrm{e}^{\omega z} \begin{pmatrix} \cos(\theta z) \\ \omega\cos(\theta z) - \theta \sin(\theta z) \end{pmatrix} + a_4 \mathrm{e}^{\omega z} \begin{pmatrix} \sin(\theta z) \\ \theta\cos(\theta z) + \omega \sin(\theta z) \end{pmatrix},$$

wherein $a_i \in \mathbb{R}$ is not all equal to zero and is uniquely determined, $i = 1, 2, 3, 4$. Hence, one has

$$I_{h,*}(z) = a_1 \mathrm{e}^{\zeta_- z} + a_2 \mathrm{e}^{\zeta_+ z} + \mathrm{e}^{\omega z} l_1(z), \tag{4.7}$$

and

$$I_{v,*}(z) = a_1 \beta_{\zeta_-} \mathrm{e}^{\zeta_- z} + a_2 \beta_{\zeta_+} \mathrm{e}^{\zeta_+ z} + \mathrm{e}^{\omega z} l_2(z), \tag{4.8}$$

with $l_1(z) = a_3 \cos(\theta z) + a_4 \sin(\theta z)$ and $l_2(z) = a_3[\omega\cos(\theta z) - \theta \sin(\theta z)] + a_4[\theta \cos(\theta z) + \omega \sin(\theta z)]$, for $z \in \mathbb{R}$. Obviously, $l_2(z) = \omega l_1(z) + l_1'(z)$.

To prove $l_1(z)$ and $l_2(z)$ change sign for $|z| \gg 1$. We first claim $l_1(z) \not\equiv 0$, $z \in \mathbb{R}$. Suppose not. If $l_1(z) \equiv 0$, then one gets $a_1 \geq 0$ and $a_2 \geq 0$ because $I_{h,*}(z) > 0$, $z \in \mathbb{R}$ and $\zeta_- < 0 < \zeta_+$. Accordingly, $I_{v,*}(z) \leq 0$, $z \in \mathbb{R}$ which is due to $l_2(z) = \omega l_1(z) + l_1'(z)$, and $\beta_{\zeta_-} < 0$, $\beta_{\zeta_+} < 0$. This contradicts the system (4.6) which implies that $l_1(z) \not\equiv 0$. That is, a_3 and a_4 are not equal to zero at the same time. Without loss of generality, letting $a_3 > 0$, thus we obtain $l_1(0) = a_3 > 0$, $l_1(\pi/\theta) = -a_3 < 0$. Therefore, $l_1(z)$ changes sign for $|z| \gg 1$. Next, we assert $l_2(z) \not\equiv 0$, $z \in \mathbb{R}$. If not, then $l_2(z) \equiv 0$. According to the facts $I_{v,*} > 0$, $z \in \mathbb{R}$, $\zeta_- < 0 < \zeta_+$, $\beta_{\zeta_-} < 0$ and $\beta_{\zeta_+} < 0$, we get $a_1 \leq 0$ and $a_2 \leq 0$. So, by (4.7), $I_{h,*}(z) < 0$ whenever $l_1(z) < 0$, contradicting (4.6). This indicates $l_2(z) \not\equiv 0$ which implies that $a_3\omega + a_4\theta$ and $a_4\omega - a_3\theta$ are not equal to zero at the same time. Supposing $a_3\omega + a_4\theta > 0$, one has $l_2(0) = a_3\omega + a_4\theta > 0$ and $l_2(\pi/\theta) = -(a_3\omega + a_4\theta) < 0$. Thus, $l_2(z)$ changes sign for $|z| \gg 1$. It follows from the above analysis and the facts $I_{h,*}(z) > 0$, $I_{v,*}(z) > 0$ that $a_1 \neq 0$ or $a_2 \neq 0$.

To address $\omega \notin \{\zeta_-, \zeta_+\}$. By the way of contradiction, assuming $\omega = \zeta_-$. From (4.7), one gets

$$I_{h,*}(z) = [a_1 + l_1(z)]\mathrm{e}^{\zeta_- z} + a_2 \mathrm{e}^{\zeta_+ z}, \quad z \in \mathbb{R}.$$

Owing to $I_{h,*}(z) > 0$ and $\zeta_- < 0 < \zeta_+$, we have $a_1 > -\min_{z\in\mathbb{R}}\{l_1(z)\} > 0$. By (4.8), then

$$I_{v,*}(z) = [a_1\beta_{\zeta_-} + l_2(z)]\mathrm{e}^{\zeta_- z} + a_2 \beta_{\zeta_+} \mathrm{e}^{\zeta_+ z}, \quad z \in \mathbb{R}.$$

Hence $a_1\beta_{\zeta_-} > -\min_{z\in\mathbb{R}}\{l_2(z)\} > 0$ because $I_{v,*}(z) > 0$ and $\zeta_- < 0 < \zeta_+$. So, $a_1 < 0 < a_1$ due to $\beta_{\zeta_-} < 0$, which is a contradiction. Similarly, one can show $\omega \neq \zeta_+$. Thus, $\omega \notin \{\zeta_-, \zeta_+\}$.

Since $l_1(z)$ changes sign when $|z| \gg 1$, we obtain $\zeta_- < \omega < \zeta_+$, $a_1 \gg 0$ and $a_2 \gg 0$ in order to ensure $I_{h,*}(z) > 0$. By (4.8) and combining $\beta_{\zeta_-} < 0$ and $\beta_{\zeta_+} < 0$, one has $I_{v,*}(z) < 0$, $|z| \gg 1$ which contradicts (4.6). In summary, in this case, either $I_{h,*}(z)$ or $I_{v,*}(z)$ changes sign.

Case 2 Matrix $\mathbb{A}$ has no complex eigenvalues.

(i) If matrix $\mathbb{A}$ has four real distinct eigenvalues, denote them as ζ_-, ζ_+, ζ_3 and ζ_4. Likely above discussions, there exist eigenvectors G_{ζ_-}, G_{ζ_+}, G_{ζ_3} and G_{ζ_4} such that

$$\begin{aligned} \begin{pmatrix} I_{h,*}(z) \\ I_{v,*}(z) \end{pmatrix} &= a_1 \mathrm{e}^{\zeta_- z} G_{\zeta_-} + a_2 \mathrm{e}^{\zeta_+ z} G_{\zeta_+} a_3 \mathrm{e}^{\zeta_3 z} G_{\zeta_3} + a_4 \mathrm{e}^{\zeta_4 z} G_{\zeta_4} \\ &= \begin{pmatrix} a_1 \mathrm{e}^{\zeta_- z} + a_2 \mathrm{e}^{\zeta_+ z} + a_3 \mathrm{e}^{\zeta_3 z} + a_4 \mathrm{e}^{\zeta_4 z} \\ a_1 \beta_{\zeta_-} \mathrm{e}^{\zeta_- z} + a_2 \beta_{\zeta_+} \mathrm{e}^{\zeta_+ z} + a_3 \beta_{\zeta_3} \mathrm{e}^{\zeta_3 z} + a_4 \beta_{\zeta_4} \mathrm{e}^{\zeta_4 z} \end{pmatrix}, \quad z \in \mathbb{R}, \end{aligned}$$

here $a_i \in \mathbb{R}$ is not all equal to zero, $i = 1, 2, 3, 4$. Because the four eigenvalues are distinct, and β_{ζ_-}, β_{ζ_+}, β_{ζ_3} and β_{ζ_4} are negative, and $\min\{\zeta_-, \zeta_+, \zeta_3, \zeta_4\} < 0 < \max\{\zeta_-, \zeta_+, \zeta_3, \zeta_4\}$, one obtains either $I_{h,*}(z_0) < 0$ or $I_{v,*}(z_0) < 0$ for some $z_0 \in \mathbb{R}$. This contradicts system (4.6).

(ii) If matrix $\mathbb{A}$ has a pair of double eigenvalues. It is not hard to see $\mathbb{A}$ cannot have two pairs of double eigenvalues. Otherwise, the characteristic equation of $\mathbb{A}$ is $Q(\zeta) = (\zeta - \zeta_-)^2(\zeta - \zeta_+)^2$. Then

$Q(0) = \zeta_-^2\zeta_+^2 > 0$ which contradicts the fact $Q(0) < 0$. Accordingly, one only needs to consider the following situations

$$Q(\zeta) = (\zeta - \zeta_3)^2(\zeta - \zeta_-)(\zeta - \zeta_+),$$

where $\zeta_- < 0 < \zeta_+$ and $\zeta_3 \notin \{\zeta_-, \zeta_+\}$. Recalling that $\mathrm{Dim}(\mathbb{G}_{\zeta_3}) = 1$, so

$$\begin{pmatrix} I_{h,*}(z) \\ I_{v,*}(z) \end{pmatrix} = a_1 \mathrm{e}^{\zeta_- z} G_{\zeta_-} + a_2 \mathrm{e}^{\zeta_+ z} G_{\zeta_+} + \mathrm{e}^{\zeta_3 z}\left[a_3 G_{\zeta_3} + a_4(z G_{\zeta_3} + G^1_{\zeta_3})\right], \quad z \in \mathbb{R},$$

where G_{ζ_3} and $G^1_{\zeta_3}$ are two linearly independent generalized eigenvectors corresponding to ζ_3.

If $a_4 = 0$, similar to (i), then there exist some $z_0 \in \mathbb{R}$ such that $I_{h,*}(z_0) < 0$ or $I_{v,*}(z_0) < 0$ which is a contradiction.

If $a_4 \neq 0$, then there are three cases:

(ii)$_1$ If $\zeta_3 \geq \zeta_+ > 0$, then $a_4 > 0$ and $a_1 > 0$ to guarantee $I_{h,*}(z) > 0$, $|z| \gg 1$. Thereby $I_{v,*}(z) < 0$ for sufficiently small $z \ll -1$ due to $a_1\beta_{\zeta_-} < 0$, which contradicts system (4.6).

(ii)$_2$ If $\zeta_3 \leq \zeta_- < 0$, then $a_4 < 0$ and $a_2 > 0$ to ensure $I_{h,*}(z) > 0$, $|z| \gg 1$. Hence, $I_{v,*}(z) < 0$ for sufficiently large $z \gg 1$ due to $a_2\beta_{\zeta_+} < 0$, contradicting system (4.6).

(ii)$_3$ If $\zeta_- < 0 < \zeta_3 < \zeta_+$, then $a_1 > 0$, $a_2 > 0$ or $a_1 > 0$, $a_2 = 0$, $a_4 > 0$ to make $I_{h,*}(z) > 0$, $|z| \gg 1$. If $\zeta_- < \zeta_3 < 0 < \zeta_+$, then $a_1 > 0$, $a_2 > 0$ or $a_1 = 0$, $a_2 > 0$, $a_4 < 0$ to ensure $I_{h,*}(z) > 0$, $|z| \gg 1$. Consequently, when $\zeta_- < \zeta_3 < \zeta_+$, one gets $I_{v,*}(z) < 0$ for sufficiently large $|z|$ according to $a_1\beta_{\zeta_-} < 0$ or $a_2\beta_{\zeta_+} < 0$. This is also a contradiction.

(iii) If matrix $\mathbb{A}$ has a triple eigenvalue. We divide it into two cases:

(iii)$_1$ If the multiplicity of eigenvalue ζ_- is three, then the characteristic equation is $Q(\zeta) = (\zeta - \zeta_-)^3(\zeta - \zeta_+)$. Since $\mathrm{Dim}(\mathbb{G}_{\zeta_-}) = 1$, we get

$$\begin{pmatrix} I_{h,*}(z) \\ I_{v,*}(z) \end{pmatrix} = a_2 \mathrm{e}^{\zeta_+ z} G_{\zeta_+} + \mathrm{e}^{\zeta_- z}\left[a_1 G_{\zeta_-} + a_3(z G_{\zeta_-} + G^1_{\zeta_-}) + a_4\left(\frac{z^2}{2} G_{\zeta_-} + z G^1_{\zeta_-} + G^2_{\zeta_-}\right)\right], \; z \in \mathbb{R},$$

where G_{ζ_-}, $G^1_{\zeta_-}$ and $G^2_{\zeta_-}$ are three linearly independent generalized eigenvectors corresponding to ζ_-. Utilizing the facts $\zeta_- < 0 < \zeta_+$ and $\beta_{\zeta_\pm} < 0$, one can similarly obtain that there are some $z_0 \in \mathbb{R}$ such that $I_{h,*}(z_0) < 0$ or $I_{v,*}(z_0) < 0$ which contradicts system (4.6).

(iii)$_2$ If the multiplicity of eigenvalue ζ_+ is three, then $Q(\zeta) = (\zeta - \zeta_+)^3(\zeta - \zeta_-)$. Similar to arguments of (iii)$_1$, we get $I_{h,*}(z)$ or $I_{v,*}(z)$ change sign for some $z_0 \in \mathbb{R}$ which is a contradiction with system (4.6).

Consequently, combining *Cases 1* and *2*, we obtain that system (1.2) has no TWS connecting E_0 and E_1^* when $\mathfrak{R}_0 > 1$, $0 < c < c_*$. This completes the proof. □

Remark 4.1. Although the idea of Theorem 4.2 comes from [4], we have further improved their methods. In addition, from Theorems 3.1, 4.1 and 4.2, it concludes that c_* is the minimal wave speed for (1.2).

5. Numerical simulations

In this section, we apply system (1.2) to the spread of dengue fever and provide some numerical simulations to verify the existence of TWS. For simplicity, we let $f_1(S_h, I_v) = \beta_1 S_h I_v$ and $f_2(S_v, I_h) = \beta_2 S_v I_h$, where β_i is positive constant and represents the transmission rate of dengue fever, $i = 1, 2$, and we take the spatial domain $[0, 100]$ and the temporal domain $[0, 400]$.

Assume (1.2) satisfies the following initial conditions

$$S_h(0, x) = \begin{cases} S_h^*, & x \in [0, 50), \\ S_h^0, & x \in [50, 100], \end{cases} \qquad I_h(0, x) = \begin{cases} I_h^*, & x \in [0, 50), \\ 0, & x \in [50, 100], \end{cases}$$

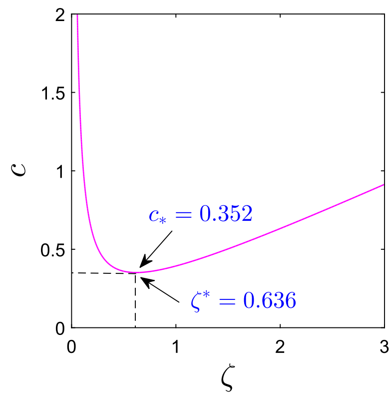

Fig. 1. The relationship between c and ζ

and

$$S_v(0,x) = \begin{cases} S_v^*, & x \in [0,50), \\ S_v^0, & x \in [50,100], \end{cases} \qquad I_v(0,x) = \begin{cases} I_v^*, & x \in [0,50), \\ 0, & x \in [50,100]. \end{cases}$$

Moreover, we take homogeneous Neumann boundary conditions for system (2.2). In view of [1,4,12,29], we assume $\Lambda = 100$, $M = 0.2$, $D_S = 0.2$, $D_I = 0.1$, $d_S = 0.5$, $d_I = 0.3$, $\mu_1 = 0.83$, $d_1 = 0.001$, $d_2 = 0.0001$, $\mu_2 = 0.002$, $\beta_1 = 0.00682$, $\beta_2 = 0.0015$, $\alpha_1 = 0.1667$. Thus, $\gamma_1 = \mu_1 + d_1 + \alpha_1 = 0.9977$ and $\gamma_2 = \mu_2 + d_2 = 0.0021$. By simple calculations, we obtain the threshold $\mathfrak{R}_0 = 7.6699 > 1$, and

$$E_0 = (120.4819, 0, 100, 0), \quad E_1^* = (68.4884, 43.2541, 2.9904, 92.3901).$$

Therefore, according to Lemma 2.1 and Theorem 3.1, there exists $c_* > 0$ such that (2.2) admits a TWS connecting E_0 and E_1^* with speed c for each $c \geq c_*$. According to Fig. 1, it can be found that the minimal wave speed c_* is 0.352.

We reveal the results in Figs. 2 and 3. Figure 2 is the corresponding contour graphs which illustrates the change of humans and mosquitoes densities. The red arrows in Fig. 2 indicate that the solution of (1.2) evolves from the disease-free equilibrium E_0 to endemic equilibrium E_1^* coinciding with Theorem 3.1. Furthermore, to present the shape of solutions more clearly, Fig. 3 depicts the cross section curves of the solution at different times. As described in Fig. 3, one can see that the TWS of (1.2) is not monotone owing to the constant recruitment and natural death in the model.

To explore the influence of parameters on the spread of the disease, we next investigate the sensitivity of c_* on parameters when $\mathfrak{R}_0 > 1$. According to Lemma 2.1, one has

$$U_2^{c_*}(\zeta^*)U_4^{c_*}(\zeta^*) - \beta_1\beta_2 S_h^0 S_v^0 = 0,$$

where $S_h^0 = \Lambda/\mu_1$, $S_v^0 = M/\mu_2$ and

$$U_2^{c_*}(\zeta^*) = D_I\zeta^{*2} - c_*\zeta^* - \gamma_1, \quad U_4^{c_*}(\zeta^*) = d_I\zeta^{*2} - c_*\zeta^* - \gamma_2.$$

Simple calculations show that

$$\frac{\partial c_*}{\partial D_I} > 0, \quad \frac{\partial c_*}{\partial d_I} > 0, \quad \frac{\partial c_*}{\partial S_h^0} > 0, \quad \frac{\partial c_*}{\partial S_v^0} > 0, \quad \frac{\partial c_*}{\partial \beta_i} > 0, \quad i = 1, 2.$$

Thus, the c_* increases monotonically with respect to Λ, M, β_i, D_I and d_I, decreases with respect to μ_i. Figure 4 illustrates the sensitivity of c_* on parameters when $\mathfrak{R}_0 > 1$ and the values of other parameters are the same as in Fig. 1. Some noteworthy phenomena are found in Fig. 4. As can be seen in Fig. 4a, c_* is an increasing function of D_I and d_I, and furthermore the effect of d_I on c_* is greater than that of D_I on c_* which implies that the diffusion of infected mosquitoes has a more significant impact on the spread

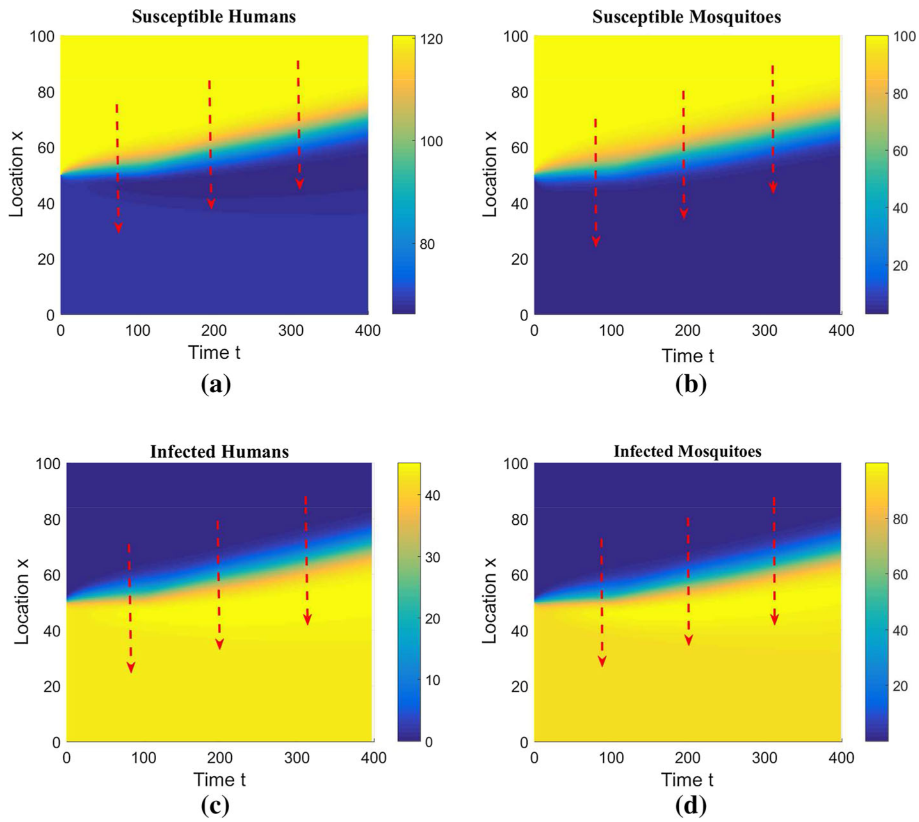

Fig. 2. The contour graph of traveling wave solution for system (1.2). **a** The evolution of S_h. **b** The evolution of S_v. **c** The evolution of I_h. **d** The evolution of I_v

of dengue fever. In Fig. 4b, it can be found that c_* increases with the increase of β_2 when β_1 is large, but c_* does not vary significantly with the increase of β_2 when β_1 is small. Note that β_1 (β_2) denotes the disease transmission rate from infectious mosquitoes (humans) to humans (mosquitoes). Figure 4b shows that the infected mosquitoes have a greater impact on the spread of dengue fever than infected people. In Fig. 4c, c_* increases with the increase of Λ when M is large, but the change of c_* is not obvious with the increase of Λ when M is small. From Fig. 4c, compared with the recruitment of individuals, the impact of the recruitment of mosquitoes is greater on the spread of dengue fever. In Fig. 4d, c_* reduces with the increase of μ_1 when μ_2 is small, but the change of c_* is not obvious with the increase of μ_1 when μ_2 is large, which means that the natural death of mosquitoes has a more obvious impact on disease transmission than the natural death of people. It can be seen from Fig. 4 that the parameters related to mosquitoes have a greater impact on minimal wave speed. Consequently, to better prevent the spread of disease, we should pay more attention to mosquito control, such as spraying insecticides and using bed nets.

6. Discussion

In this work, we proposed a reaction–diffusion mosquito-borne epidemic model with general incidence and constant recruitment, and discussed the existence and nonexistence of nontrivial TWS for this model.

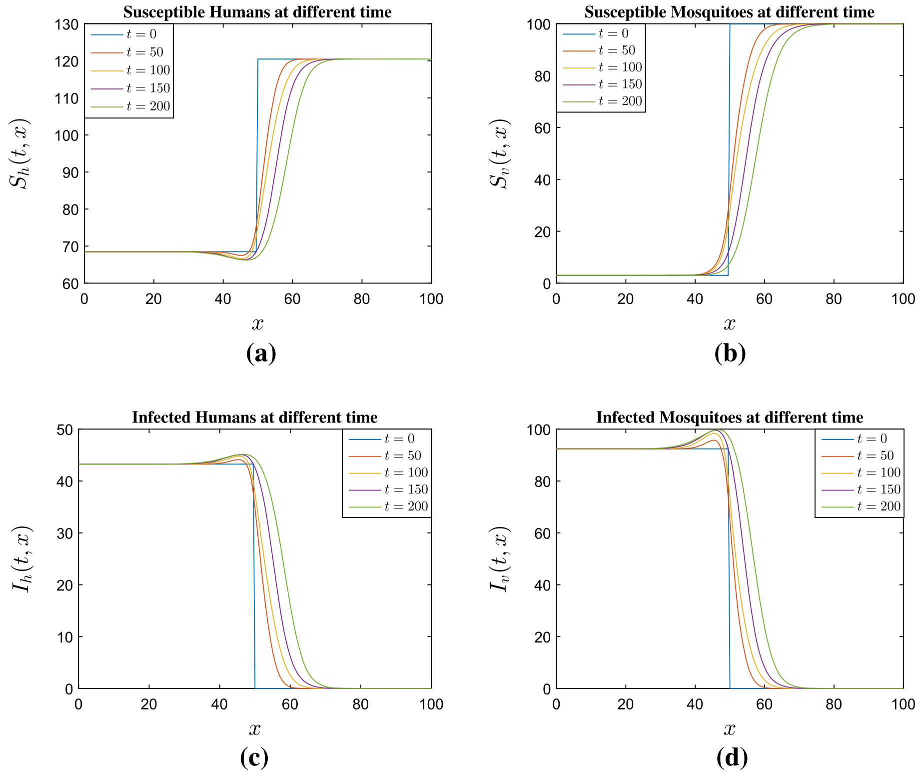

FIG. 3. The cross section curves of traveling wave solution for (1.2) at different times. **a** The curve of S_h. **b** The curve of S_v. **c** The curve of I_h. **d** The curve of I_v

Specifically, for the case of $\Re_0 > 1$ and $c \geq c_*$, the suitable sub- and super-solutions were constructed by means of the smallest positive eigenvalue of the characteristic equation, and the existence of solutions for the truncated system was obtained by using the fixed-point theorem. Then it was proved that there exists a nontrivial TWS of model (1.2) satisfying (2.3) with the help of limiting arguments. The convergence of TWS at positive infinity was showed by constructing a Lyapunov functional. Next, the nonexistence of nontrivial TWS when $\Re_0 \leq 1$ and $c > 0$ was established by utilizing contradicting approach. For the case of $\Re_0 > 1$ and $0 < c < c_*$, by illustrating that $I_h(\cdot)$ or $I_v(\cdot)$ will change sign at some points, we proved that the system (1.2) has no nontrivial TWS connecting E_0 and E_1^*. We should note that the mathematical analysis of the process was complicated, but easier to understand than the method of Laplace transform.

In order to better elaborate the theoretical results, we applied system (1.2) to investigate the spread of dengue fever. We provided numerical simulations to verify the theoretical results of this paper (see Figs. 2, 3). By discussing the sensitivity of c_* on parameters, the combined effects of parameters on c_* were analyzed (see Fig. 4). It can be known that (1) the c_* can be reduced by decreasing the diffusion of infectious humans and mosquitoes (see Fig. 4a); (2) by adopting relevant measures, such as spraying insecticides and using bed nets, the biting rate can be reduced. As a result, the c_* can be reduced (see Fig. 4b–d).

As we all know, the transmission of many infectious diseases, including mosquito-borne diseases, is deeply affected by environment temperature [19], and temperature can be characterized by time periodicity. Hence, it is reasonable to incorporate time periodicity into the modeling of infectious diseases.

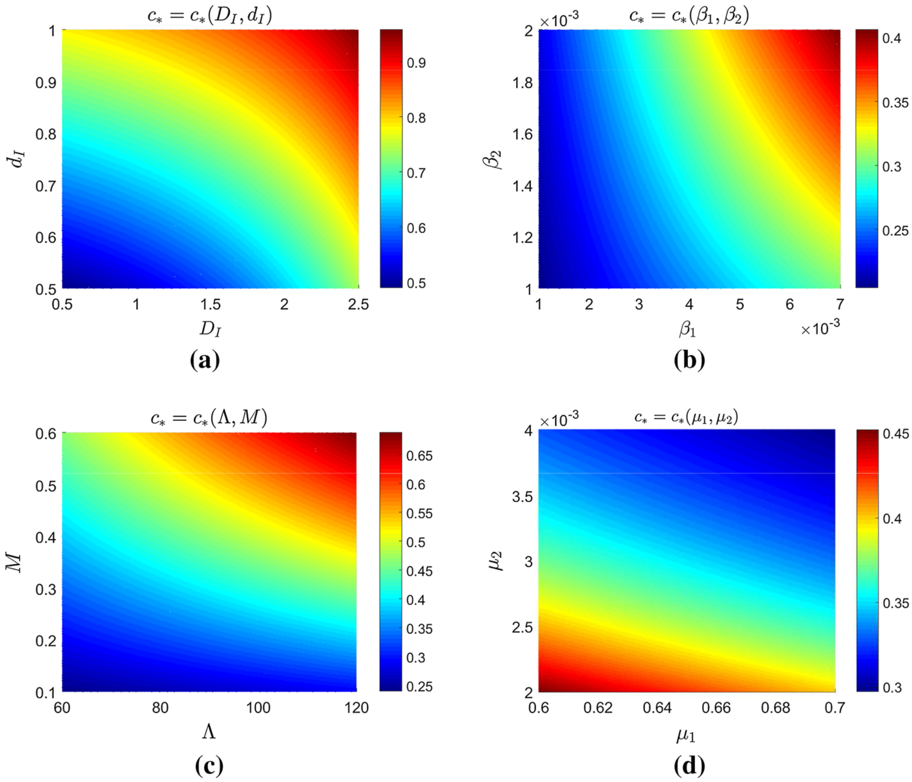

FIG. 4. The sensitivity of c_* on parameters for system (1.2). **a** The sensitivity of c_* on D_I and d_I. **b** The sensitivity of c_* on β_1 and β_2. **c** The sensitivity of c_* on Λ and M. **d** The sensitivity of c_* on μ_1 and μ_2

In this case, it is interesting and important to study the periodic traveling wave solutions of epidemic models [8,28,31,34]. However, there are some challenges in exploring the periodic traveling wave solutions of mosquito-borne disease models. We leave these issues for future study.

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