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Spatiotemporal dynamics in the single population model with memory-based diffusion and nonlocal effect

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Abstract

To incorporate spatial memory and nonlocal effect of animal movements, we propose and investigate the spatiotemporal dynamics of the single population model with memory-based diffusion and nonlocal reaction. We first study the stability of a positive equilibrium and the steady state bifurcation induced by diffusion and nonlocality. We then investigate the impact of the averaged memory period on stability and bifurcation, and show that the combination of the averaged memory period and the diffusion can lead to the occurrence of Turing-Hopf and double Hopf bifurcations. The paper originally derives the normal form theory for Turing-Hopf bifurcation in the general reaction-diffusion equation with memory-based diffusion and nonlocal reaction. This novel algorithm can be widely used to classify the spatiotemporal dynamics near the Turing-Hopf bifurcation point. Finally, we apply the obtained results to a model proposed by Britton and numerically illustrate the spatiotemporal patterns induced by Hopf, Turing-Hopf and double Hopf bifurcations. Stable spatially homogeneous/nonhomogeneous periodic solutions, homogeneous/nonhomogeneous steady states and the transition from one of these solutions to another are provided in this paper. We additionally acquire the coexistence of two stable spatially nonhomogeneous steady states or two spatially nonhomogeneous periodic solutions near the Turing-Hopf bifurcation point.
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1. Introduction

Recently, spatial memory and cognition of animals has drawn much attention in the mathematical modeling of animal movements. The incorporation of spatial memory into our existing partial differential equation models for animal movements is pivotal but challenging as summarized in a recent review paper [12]. To incorporate spatial memory and cognition in a self-contained way, Shi et al. [24] proposed the following delayed diffusion model via a modified Fick's law:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + d_2 div(u \nabla u_\tau) + f(u), & x \in \Omega, t > 0, \\ \frac{\partial u(x,t)}{\partial n} = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ describes the population density at the spatial location x and at time t, $u_\tau = u(x, t-\tau)$ and Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$. Here, d_1 and d_2 are the coefficients of Fickian diffusion and the memory-based diffusion, respectively, the time delay $\tau \geq 0$ represents the averaged memory period, and $f(u)$ is the reaction term and describes the chemical reaction or biological birth/death. Different from the classical reaction-diffusion model, the main feature of (1.1) is that there is a directed movement toward the negative or positive gradient of the density distribution function at past time.

In model (1.1), the reaction term $f(u)$ only depends on the local density at the spatial location x. In recent years, however, it has become recognized that the birth/growth/death rates in the reaction term of individuals located at a spatial point x can heavily depend on densities in other spatial points. For this case, the model is called the population model with nonlocal interactions. The nonlocal nature of this class of models is more realistic than the local models. The rich dynamics in the nonlocal models have been investigated by many researchers [1,3–6,8,13,15–17,21–23,25,29]. We can consider the total available resource, then the averaged density over space matters as we propose in model (1.3). As shown in [15], the most straightforward way of introducing nonlocal effects is to replace the term $f(u)$ by $f(u, \widehat{u})$, where

$$\widehat{u} = \int_\Omega G(x,y) u(y,t) dy, \tag{1.2}$$

where $G(x,y)$ is some reasonable kernel and Ω is the spatial domain. If $G(x,y)$ is a Dirac delta function at x, the nonlocal model reduces to be local. Here, we take $G(x,y) = 1/(\text{vol } \Omega)$ such that the nonlocal effect does not affect the positive equilibrium of (1.1). This kernel function means that $\widehat{u}$ is taken as the mean value of u in the space interval $[0, \ell\pi]$. More recently, for this mean value kernel, the influence of the nonlocal term on the spatiotemporal dynamics of the diffusive predator-prey model has been recently investigated in [9,10,31] and we found that the nonlocal term is a key factor to generate the spatially nonhomogeneous spatiotemporal dynamics.

In this paper, we introduce the nonlocal effect into (1.1) and investigate its spatiotemporal dynamics. For simplicity of notations, we consider the following nonlocal version of (1.1) on one-dimensional spatial domain $\Omega = (0, \ell\pi)$ with $\ell \in \mathbb{R}^+$ and $G(x,y) = 1/(\ell\pi)$, i.e.,

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 u_{xx} + d_2 (u u_x(x, t-\tau))_x + f(u, \widehat{u}), & 0 < x < \ell\pi, t > 0, \\ u_x(0,t) = u_x(\ell\pi, t) = 0, \end{cases} \tag{1.3}$$

where $\widehat{u} = \frac{1}{\ell\pi} \int_0^{\ell\pi} u(y,t) dy$.

For the local model (1.1), the authors have shown that the stability of the positive equilibrium fully depends on the reaction term and the relationship between the two diffusion coefficients but is independent of the time delay. In this paper, we find that the nonlocal model (1.3) has far richer dynamics and the role of spatial memory becomes more significant. Because of the nonlocal form, the combination of the memory-based diffusion coefficient d_2 and the averaged memory period τ can yield codimension-two bifurcation phenomena such as Turing-Hopf and double Hopf bifurcations, which lead to the occurrence of stable spatially homogeneous/nonhomogeneous periodic solutions, stable spatially nonhomogeneous quasi-periodic solutions, homogeneous/nonhomogeneous steady states, and the transition from one of these solutions to another. It is well known that there is no Turing bifurcation in the classical scalar reaction-diffusion equation. The codimension-two Turing-Hopf bifurcation in the two species population model has been widely found in the literature [2,7,11,20,26,27,30,32]. Nevertheless, to our best knowledge, this is the first paper in acquiring the Turing-Hopf bifurcation in a scalar population model, caused by the integrative effect of spatial memory and nonlocal mechanism.

The paper is organized as follows. In Section 2, we investigate the distribution of characteristic roots and derive the conditions for the stability of the positive equilibrium and the existence of the Hopf bifurcation and Turing-Hopf bifurcation. In Section 3, we derive the algorithm of normal form of the Turing-Hopf bifurcation for the general single species model with random and memory-based diffusion terms. In Section 4, we apply the obtained theoretical results to a single biological population proposed by Britton [5,6] and study the dynamical classification near the Turing-Hopf bifurcation point. Finally, we discuss our theoretical results and its general applicability to other models as well as future directions in Section 5.

Throughout the paper, $\mathbb{N} = \{1, 2, \cdots\}$ represents the set of all positive integers, and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ represents the set of all non-negative integers.

2. Stability and bifurcation analysis

Assume that u_* is the positive equilibrium of system (1.3), i.e., $f(u_*, u_*) = 0$. The linearized system of (1.3) at u_* is

$$\frac{\partial u(x,t)}{\partial t} = d_1 u_{xx} + d_2 u_* u_{xx}(x, t-\tau) + Au + B\widehat{u}, \tag{2.1}$$

where

$$A = \left.\frac{\partial f(u, \widehat{u})}{\partial u}\right|_{(u_*, u_*)}, \quad B = \left.\frac{\partial f(u, \widehat{u})}{\partial \widehat{u}}\right|_{(u_*, u_*)}.$$

It is well known that the eigenvalue problem

$$\begin{cases} -\gamma''(x) = \mu\gamma(x), & x \in (0, \ell\pi), \\ \gamma'(0) = \gamma'(\ell\pi) = 0, \end{cases} \tag{2.2}$$

has eigenvalues $\mu_n = (n/\ell)^2$, $n \in \mathbb{N}_0$, with corresponding eigenfunctions $\cos\left(\frac{n}{\ell}x\right)$. Then, letting $u(x,t) = e^{\lambda_n t}\cos\left(\frac{n}{\ell}x\right)$ and substituting it into (2.1), one can obtain the characteristic equation

$$\begin{cases} \lambda_0 = A + B, & n = 0, \\ \lambda_n = -\left(d_1 + d_2 u_* e^{-\lambda_n \tau}\right) \mu_n + A, & n \in \mathbb{N}. \end{cases} \tag{2.3}$$

In the following, we always assume that $A + B < 0$, which implies that the positive equilibrium u_* is asymptotically stable in the absence of diffusions, and investigate the influence of the diffusion coefficients d_1, d_2 and the delay τ on the stability of the positive equilibrium u_*.

2.1. Stability and bifurcation for the case without delay

For Eq. (2.3) with $\tau = 0$, it is easy to see that

$$\lambda_n \begin{cases} < 0, & d_2 > -\frac{1}{u_*} d_1 + \frac{A\ell^2}{u^* n^2}, \\ = 0, & d_2 = -\frac{1}{u_*} d_1 + \frac{A\ell^2}{u^* n^2}, \\ > 0, & d_2 < -\frac{1}{u_*} d_1 + \frac{A\ell^2}{u^* n^2}. \end{cases} \tag{2.4}$$

In addition, notice that $d_1 > 0$ and $d_2 \in \mathbb{R}$. Then the following lemma follows immediately.

Lemma 2.1. *Assume that $A + B < 0$ and $\tau = 0$. Define*

$$D^+ = \left\{ (d_1, d_2) \,\middle|\, d_1 > 0,\ d_2 > -\frac{1}{u_*} d_1 + \frac{A\ell^2}{u^*} \right\}, \tag{2.5}$$

$$D^- = \left\{ (d_1, d_2) \,\middle|\, d_1 > 0,\ d_2 > -\frac{1}{u_*} d_1 \right\}, \tag{2.6}$$

$$D_u^+ = \left\{ (d_1, d_2) \,\middle|\, d_1 > 0,\ d_2 < -\frac{1}{u_*} d_1 + \frac{A\ell^2}{u^*} \right\}, \tag{2.7}$$

and

$$D_u^- = \left\{ (d_1, d_2) \,\middle|\, d_1 > 0,\ d_2 < -\frac{1}{u_*} d_1 \right\}. \tag{2.8}$$

(*I*) *If $A > 0$, then when $(d_1, d_2) \in D^+$, $\lambda_n < 0$ for any $n \in \mathbb{N}_0$, and when $(d_1, d_2) \in D_u^+$, there exists some $n \in \mathbb{N}$ such that $\lambda_n > 0$;*
(*II*) *If $A \le 0$, then when $(d_1, d_2) \in D^-$, $\lambda_n < 0$ for any $n \in \mathbb{N}_0$, and when $(d_1, d_2) \in D_u^-$, there exists some $n \in \mathbb{N}$ such that $\lambda_n > 0$.*

It follows from (2.4) that taking d_2 as a parameter, $\lambda_n = 0$ if and only if

$$d_2 = d_{2,n}^* \triangleq -\frac{1}{u_*} d_1 + \frac{A\ell^2}{u_* n^2}. \tag{2.9}$$

We treat λ_n as a function of d_2, and it satisfies $\lambda_n(d_2)|_{d_2 = d_{2,n}^*} = 0$. Then we have the following transversality condition

$$\left.\frac{d\left(\lambda_n(d_2)\right)}{dd_2}\right|_{d_2=d_{2,n}^*} = -u_*\left(\frac{n}{\ell}\right)^2 < 0. \tag{2.10}$$

Thus, by Lemma 2.1 and the transversality condition (2.10), we have the following results.

Theorem 2.1. *Assume that $A+B<0$, $\tau=0$ and $d_{2,n}^*$ is defined by (2.9).*

- (I) *If $A>0$, then for fixed $d_1>0$, u_* is asymptotically stable for $d_2>d_{2,1}^*$ and unstable for $d_2<d_{2,1}^*$. Moreover, steady state bifurcations occur at $d_2=d_{2,n}^*$;*
- (II) *If $A\le 0$, then for fixed $d_1>0$, u_* is asymptotically stable for $d_2>-\frac{1}{u_*}d_1$ and unstable for $d_2<-\frac{1}{u_*}d_1$. Moreover, steady state bifurcations occur at $d_2=d_{2,n}^*$.*

We would like to mention that the results in Theorem 2.1 is also known from other papers, such as Britton [6] and Shi et al. [25].

Remark 2.1. Notice that there is no complex roots for (2.3) with $\tau=0$. For the distribution of the positive real roots, it follows from Eq. (2.4) that if $A>0$, then Eq. (2.3) has n positive real roots for $d_{2,n+1}^*<d_2<d_{2,n}^*$ and infinitely many positive real roots for $d_2\le -\frac{1}{u_*}d_1$, while if $A\le 0$, then Eq. (2.3) has infinitely many positive real roots for $d_2<-\frac{1}{u_*}d_1$.

2.2. *The effect of delay on the stability and bifurcation*

According to Theorem 2.1, we know that for $\tau=0$, if $A>0$, then u_* is asymptotically stable for $(d_1,d_2)\in D^+$ and unstable for $(d_1,d_2)\in D_u^+$, while if $A\le 0$, then u_* is asymptotically stable for $(d_1,d_2)\in D^-$ and unstable for $(d_1,d_2)\in D_u^-$, where D^+, D^-, D_u^+ and D_u^- are defined by (2.5), (2.6), (2.7) and (2.8), respectively.

We first define

$$p(\lambda_n)=\lambda_n+\left(d_1+d_2u_*e^{-\lambda_n\tau}\right)\left(\frac{n}{\ell}\right)^2-A,\ \lambda_n\in\mathbb{R}. \tag{2.11}$$

Then, for $n\in\mathbb{N}$, Eq. (2.3) is equivalent to $p(\lambda_n)=0$. It is easy to verify that

$$p(0)\begin{cases} >0, & \text{for } d_2>d_{2,n}^*,\\ =0, & \text{for } d_2=d_{2,n}^*,\\ <0, & \text{for } d_2<d_{2,n}^*,\end{cases} \tag{2.12}$$

where $d_{2,n}^*$ is defined by (2.9), and,

$$\frac{dp(\lambda_n)}{d\lambda_n}=1-\frac{\tau d_2u_*n^2}{\ell^2}e^{-\lambda_n\tau}\begin{cases} >0, & \text{for } d_2\le 0,\ \text{or } d_2>0 \text{ and } \lambda_n>\frac{1}{\tau}\ln\frac{\tau d_2u_*n^2}{\ell^2},\\ =0, & \text{for } d_2>0 \text{ and } \lambda_n=\frac{1}{\tau}\ln\frac{\tau d_2u_*n^2}{\ell^2},\\ <0, & \text{for } d_2>0 \text{ and } \lambda_n<\frac{1}{\tau}\ln\frac{\tau d_2u_*n^2}{\ell^2}.\end{cases} \tag{2.13}$$

It therefore follows that when $d_2<d_{2,n}^*$, $p(\lambda_n)=0$ has a unique positive real root. Noting that $\lambda_n=0$ is a root of $p(\lambda_n)=0$ if and only if $d_2=d_{2,n}^*$, we have the following result.

Lemma 2.2. *Assume that $A + B < 0$ and $\tau > 0$.*

(I) *If $A > 0$, then when $(d_1, d_2) \in D^+$, Eq. (2.3) has no zero roots, and Eq. (2.3) has at least n positive real roots for $d^*_{2,n+1} < d_2 < d^*_{2,n}$ and infinitely many positive real roots for $d_2 \le -\frac{1}{u_*}d_1$;*

(II) *If $A \le 0$, then when $(d_1, d_2) \in D^-$, Eq. (2.3) has no zero roots, and when $(d_1, d_2) \in D_u^-$, Eq. (2.3) has infinitely many positive real roots for $d_2 < -\frac{1}{u_*}d_1$.*

According to Lemma 2.2 and the definitions D_u^+ and D_u^- defined by (2.5), (2.6), respectively, we find that when $\tau > 0$, u_* is still unstable provided that $A > 0$ and $(d_1, d_2) \in D_u^+$, or provided that $A \le 0$ and $(d_1, d_2) \in D_u^-$. In the following, we investigate the influence of the time delay τ on the stability for the following two cases: (i) $A > 0$ and $(d_1, d_2) \in D^+$; (ii) $A \le 0$ and $(d_1, d_2) \in D^-$.

It is well known that the stability of u_* changes only when there exists an eigenvalue with zero real part with the increasing of the time delay. By Lemma 2.2, we know that for cases (i) and (ii), there are no zero roots for $\tau \ge 0$. Next, we investigate the distribution of purely imaginary roots of Eq. (2.3) for $\tau > 0$.

Letting

$$\tau_{n,j} = \frac{1}{\omega_n}\left\{\arccos\left(\frac{A - d_1(n/\ell)^2}{d_2u_*(n/\ell)^2}\right) + 2j\pi\right\},\ n \in \mathbb{N},\ \ j \in \mathbb{N}_0, \tag{2.14}$$

where

$$\omega_n = \sqrt{\left((d_2u_* - d_1)(n/\ell)^2 + A\right)\left((d_2u_* + d_1)(n/\ell)^2 - A\right)}, \tag{2.15}$$

we have the following results.

Lemma 2.3. *For $d_1 > 0$ and $A + B < 0$, assume that $\tau_{n,j}$ and ω_n are defined by (2.14) and (2.15), respectively. Define*

$$D_0^+ = \left\{(d_1, d_2) \,|\, d_1 > A\ell^2, -\frac{d_1}{u_*} + \frac{A\ell^2}{u_*} < d_2 \le \frac{d_1}{u_*} - \frac{A\ell^2}{u_*}\right\} \tag{2.16}$$

and

$$D_0^- = \left\{(d_1, d_2) \,|\, d_1 > 0, -\frac{d_1}{u_*} < d_2 \le \frac{d_1}{u_*}\right\}. \tag{2.17}$$

(I) *For $A > 0$,*

(i) *when $(d_1, d_2) \in D_0^+$, all roots of Eq. (2.3) have negative real parts for any $\tau \ge 0$;*

(ii) *when either*

$$d_2 > \max\left\{\frac{1}{u_*}d_1 - \frac{A\ell^2}{u_*n^2}, -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_*n^2}\right\}$$

or

$$d_2 < \min\left\{\frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2}, -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_* n^2}\right\},$$

Eq. (2.3) *has a pair of purely imaginary roots* $\pm i\omega_n$ *at* $\tau = \tau_{n,j}$.

(*II*) *For* $A < 0$,

(*i*) *When* $(d_1, d_2) \in D_0^-$, *all roots of Eq.* (2.3) *have negative real parts for any* $\tau \geq 0$;

(*ii*) *When either* $d_2 < -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_* n^2}$ *or* $d_2 > \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2}$, *Eq.* (2.3) *has a pair of purely imaginary roots* $\pm i\omega_n$ *at* $\tau = \tau_{n,j}$.

(*III*) *For* $A = 0$,

(*i*) *When* $(d_1, d_2) \in D_0^-$, *all roots of Eq.* (2.3) *have negative real parts for any* $\tau \geq 0$;

(*ii*) *When either* $d_2 < -\frac{1}{u_*}d_1$ *or* $d_2 > \frac{1}{u_*}d_1$, *Eq.* (2.3) *has a pair of purely imaginary roots* $\pm i\omega_n$ *at* $\tau = \tau_{n,j}$ *for any* $n \in \mathbb{N}$;

(*iii*) *When* $d_2 = -\frac{1}{u_*}d_1$, *Eq.* (2.3) *has zero root* $\omega_n = 0$ *for any* $n \in \mathbb{N}$.

Proof. Letting $\lambda_n = i\omega_n$ $(\omega_n > 0)$ be the root of (2.3) and substituting it into (2.3) yields

$$\begin{cases} A - d_1(n/\ell)^2 = d_2 u_* (n/\ell)^2 \cos(\omega_n \tau), \\ \omega_n = d_2 u_* (n/\ell)^2 \sin(\omega_n \tau), \end{cases} \tag{2.18}$$

from which we have

$$\omega_n^2 = \left((d_2 u_* - d_1)(n/\ell)^2 + A\right)\left((d_2 u_* + d_1)(n/\ell)^2 - A\right). \tag{2.19}$$

For $A > 0$, when $d_2 < \frac{d_1}{u_*} - \frac{A\ell^2}{u_*}$, we have $d_2 u_* - d_1 + A\ell^2 < 0$ and then $d_2 u_* - d_1 < 0$. Thus, $(d_2 u_* - d_1)(n/\ell)^2 + A \leq (d_2 u_* - d_1)(1/\ell)^2 + A = \frac{1}{\ell^2}\left(d_2 u_* - d_1 + A\ell^2\right) < 0$ for any $n \in \mathbb{N}$. Similarly, it is easy to verify that if $d_2 > -\frac{d_1}{u_*} + \frac{A\ell^2}{u_*}$, then $(d_2 u_* + d_1)(n/\ell)^2 - A > 0$ for any $n \in \mathbb{N}$. Thus, when $A > 0$ and $-\frac{d_1}{u_*} + \frac{A\ell^2}{u_*} < d_2 < \frac{d_1}{u_*} - \frac{A\ell^2}{u_*}$, for any $n \in \mathbb{N}$,

$$\left((d_2 u_* - d_1)(n/\ell)^2 + A\right)\left((d_2 u_* + d_1)(n/\ell)^2 - A\right) < 0. \tag{2.20}$$

In addition, it is easy to verify that for $A > 0$ and $d_2 = \frac{d_1}{u_*} - \frac{A\ell^2}{u_*}$, $\omega_1 = 0$ and (2.20) holds for $n = 2, 3, \cdots$. Thus, when $A > 0$ and $(d_2, d_1) \in D_0^+$, Eq. (2.3) has no purely imaginary roots. This, together with Lemma 2.1, implies that all roots of Eq. (2.3) have negative real parts for any $\tau \geq 0$.

Notice that $(d_2 u_* - d_1)(n/\ell)^2 + A > 0$ if and only if $d_2 > \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2}$, and $(d_2 u_* + d_1)(n/\ell)^2 - A > 0$ if and only if $d_2 > -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_* n^2}$. In addition, if $A > 0$, then $\frac{A\ell^2}{n^2} > 0$ and $\frac{A\ell^2}{n^2}$ is decreasing as n increases. Therefore, by (2.19), we can conclude that for $A > 0$ and fixed $n \in \mathbb{N}$, when either $d_2 < 0$ and $d_2 < \min\left\{\frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2}, -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_* n^2}\right\}$, or $d_2 > 0$ and $d_2 > \max\left\{\frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2}, -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_* n^2}\right\}$, we have

$$\left((d_2 u_* - d_1)(n/\ell)^2 + A\right)\left((d_2 u_* + d_1)(n/\ell)^2 - A\right) > 0, \text{ for fixed } n \in \mathbb{N},$$

which, together with (2.18), implies that Eq. (2.3) has a pair of purely imaginary roots $\pm i\omega_n$ at $\tau = \tau_{n,j}$. This completes the proof of (I).

For $A < 0$, notice that $\frac{A\ell^2}{n^2} < 0$ and $\frac{A\ell^2}{n^2}$ is increasing as n increases and $\lim_{n\to\infty} \frac{A\ell^2}{n^2} = 0$. Then the proof is similar to the case of $A > 0$ and we omit it here.

For $A = 0$, the conclusion immediately follows from (2.4) and (2.15). □

When (d_1, d_2) is located on the boundary of the stable regions D^+ and D^- for the case of $\tau = 0$, we have the following results on the distribution of characteristic roots with zero real parts.

Lemma 2.4. *For $d_1 > 0$ and $A + B < 0$, assume that D^+, D^-, $\tau_{n,j}$ and ω_n are defined by (2.5), (2.6), (2.14) and (2.15), respectively.*

- (I) *For $A > 0$ and (d_1, d_2) locating on the boundary of the region D^+, i.e., $d_2 = -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_*}$, then when $\frac{A\ell^2((n+1)^2+1)}{2(n+1)^2} \le d_1 < \frac{A\ell^2(n^2+1)}{2n^2}$ with $n = 2, 3, \cdots$, (2.3) has one zero root $\lambda_1 = 0$ and a pair of purely imaginary roots $\pm i\omega_k$ at $\tau = \tau_{k,j}$, $k = 2, \cdots, n$, $j \in \mathbb{N}_0$;*
- (II) *For $A < 0$ and (d_1, d_2) locating on the boundary of the region D^-, i.e., $d_2 = -\frac{1}{u_*}d_1$, (2.3) has no roots with zero real parts for any $\tau \ge 0$.*

Proof. By (2.3) and (2.4), it is easy to see that for $A > 0$, when $d_2 = -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_*}$, $\lambda_1 = 0$ is always the root of (2.3) for any $\tau \ge 0$. Notice that the straightlines $d_2 = -\frac{1}{u_*}d_1 + \frac{A\ell^2}{u_*}$ and $d_2 = \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2}$ intersect at $d_1 = \frac{A\ell^2(n^2+1)}{2n^2}$. This, together with (ii) of (I) in Lemma 2.3, completes the proof of (I).

By (2.3) and (2.4), when $d_2 = -\frac{1}{u_*}d_1$, $\lambda_n = 0$ is not a root of (2.3) for any $n \in \mathbb{N}_0$. This, together with (ii) of (II) in Lemma 2.3, completes the proof of (II). □

Assuming that $\lambda_n = \eta_n(\tau) + i\omega_n(\tau)$ is a root of (2.3) such that $\eta_n\left(\tau_{n,j}\right) = 0$ and $\omega_n\left(\tau_{n,j}\right) = \omega_n$, it follows from (2.3) that

$$\left(\frac{d\lambda_n}{d\tau}\right)^{-1} = -\frac{\tau}{\lambda_n} + \frac{e^{\lambda_n \tau}}{\lambda_n d_2 u_* (n/\ell)^2},$$

which, together with (2.18), leads to

$$\mathrm{Re}\left(\left(\frac{d\lambda_n}{d\tau}\right)^{-1}_{\tau=\tau_{n,j}}\right) = \frac{1}{\left(d_2 u_* (n/\ell)^2\right)^2} > 0.$$

Noticing that

$$\mathrm{sgn}\left\{\left.\frac{d\mathrm{Re}\lambda_n}{d\tau}\right|_{\tau=\tau_{n,j}}\right\} = \mathrm{sgn}\left\{\mathrm{Re}\left(\left(\frac{d\lambda_n}{d\tau}\right)^{-1}_{\tau=\tau_{n,j}}\right)\right\},$$

we have the following transversality condition

$$\left.\frac{d\mathrm{Re}\lambda_n}{d\tau}\right|_{\tau=\tau_{n,j}} > 0. \tag{2.21}$$

In terms of Lemmas 2.1, 2.3 and 2.4 and the transversality condition (2.21), and noticing that

$$\lim_{n\to\infty} \tau_{n,0} = 0, \tag{2.22}$$

the following results hold.

Theorem 2.2. *For $d_1 > 0, d_2 \in \mathbb{R}$, $A + B < 0$, assume that $D^+, D^-, D_u^+, D_u^-, D_0^+, D_0^-, d_{2,n}^*$, $\tau_{n,j}$ and ω_n are defined by (2.5), (2.6), (2.7), (2.8), (2.16), (2.17), (2.9) (2.14) and (2.15), respectively. Define*

$$D_n^+ = \left\{(d_1, d_2) \middle| \ \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2} < d_2 \le \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_*(n+1)^2}\right\} \cap D^+, \ n \in \mathbb{N},$$

$$D_c^s = \left\{(d_1, d_2) \middle| \ \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_*} < d_2 < \frac{1}{u_*}d_1\right\} \cap D^+,$$

$$D_\infty^+ = \left\{(d_1, d_2) \middle| \ d_2 \ge \frac{1}{u_*}d_1\right\} \cap D^+,$$

and

$$D_\infty^- = D^- \setminus D_0^-, \ \ D_1^- = \left\{(d_1, d_2) \middle| \, d_2 > \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_*}, \ d_1 > 0\right\},$$

$$D_n^- = \left\{(d_1, d_2) \middle| \, \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_* n^2} < d_2 \le \frac{1}{u_*}d_1 - \frac{A\ell^2}{u_*(n-1)^2}, \ d_1 > 0\right\}, \ n = 2, 3, \cdots.$$

(*I*) *For $A > 0$,*
- (*i*) *when $(d_1, d_2) \in D_u^+$, u_* is unstable for $\tau \ge 0$;*
- (*ii*) *when $(d_1, d_2) \in D_\infty^+$, u_* is unstable for $\tau > 0$;*
- (*iii*) *when $(d_1, d_2) \in D_0^+$, u_* is asymptotically stable for $\tau \ge 0$;*
- (*iv*) *when $(d_1, d_2) \in D_c^s$, there exists some $n \in \mathbb{N}$ such that $(d_1, d_2) \in D_n^+$ and u_* is asymptotically stable for $\tau \in [0, \tau_*)$ and unstable for $\tau \in (\tau_*, +\infty)$, and (1.3) undergoes Hopf bifurcation at $\tau = \tau_{k,j}$ with $k = 1, \cdots, n$, $j \in \mathbb{N}_0$, where*

$$\tau_* = \min\{\tau_{1,0}, \cdots, \tau_{n,0}\};$$

- (*v*) *for $\frac{A\ell^2((n+1)^2+1)}{2(n+1)^2} \le d_1 < \frac{A\ell^2(n^2+1)}{2n^2}$ with $n = 2, 3, \cdots$, (1.3) undergoes Turing-Hopf bifurcation at $(\tau, d_2) = \left(\tau_{k,j}, d_{2,1}^*\right)$, with $k = 2, \cdots, n$, $j \in \mathbb{N}_0$.*

(*II*) *For $A \le 0$,*
- (*i*) *when $(d_1, d_2) \in D_u^-$, u_* is unstable for $\tau \ge 0$;*
- (*ii*) *when $(d_1, d_2) \in D_\infty^-$, u_* is unstable for $\tau > 0$;*
- (*iii*) *when $(d_1, d_2) \in D_0^-$, u_* is asymptotically stable for $\tau \ge 0$;*

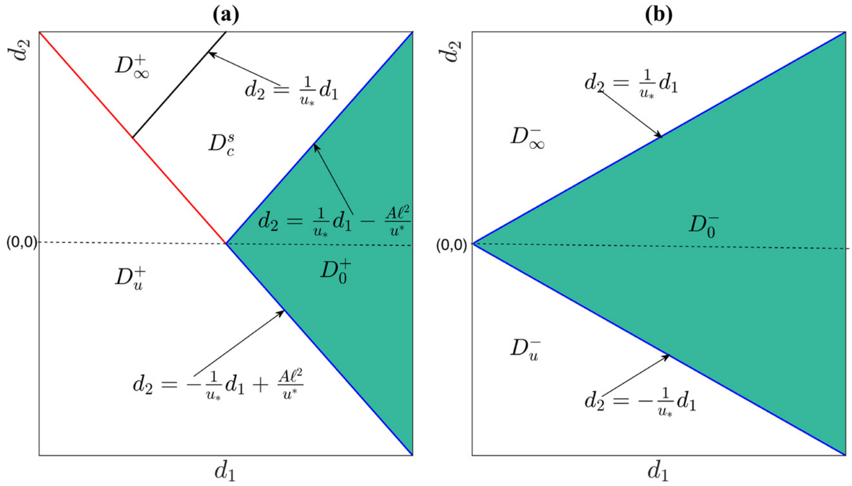

Fig. 1. Stability regions and bifurcation curves. (a) $A > 0$. When $(d_1, d_2) \in D_0^+$, u_* is stable for $\tau \geq 0$; when $(d_1, d_2) \in D_c^s$, u_* is stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$; when $(d_1, d_2) \in D_\infty^+$, u_* is unstable for $\tau > 0$; when $(d_1, d_2) \in D_u^+$, u_* is unstable for $\tau \geq 0$. (b) $A \leq 0$. When $(d_1, d_2) \in D_0^-$, u_* is asymptotically stable for $\tau \geq 0$; when $(d_1, d_2) \in D_\infty^-$, u_* is unstable for $\tau > 0$; when $(d_1, d_2) \in D_u^-$, u_* is unstable for $\tau \geq 0$.

(*iv*) *for fixed* $n \in \mathbb{N}$ *and* $(d_1, d_2) \in D_n^-$, (1.3) *undergoes Hopf bifurcation at* $\tau = \tau_{k,j}$ *with* $k = n, n+1, \cdots,\ j \in \mathbb{N}_0$.

According to Theorem 2.2, the influence of the delay on the stability of the positive equilibrium u_* of system (1.3) is illustrated in Fig. 1.

Remark 2.2. Using Lemma 2.3, together with the transversality condition (2.21) and the limit (2.22), we find that when $\tau > 0$, Eq. (2.3) has infinitely complex roots with positive real parts for $A > 0$ and $(d_1, d_2) \in D_\infty^+$, or $A \leq 0$ and $(d_1, d_2) \in D_\infty^-$. Moreover, if $A > 0$ and $(d_1, d_2) \in D_c^s$, then all the roots of Eq. (2.3) have negative real parts for $0 \leq \tau < \tau_*$, while Eq. (2.3) has finite complex roots with positive real parts for fixed $\tau > \tau_*$.

It is shown in [24] that the eigenvalue problem has either no such eigenvalues, or infinitely many such eigenvalues. However, for the current model, it is possible that the multiplicity of eigenvalues with positive real parts is finite except the cases in [24].

3. Normal form of Turing-Hopf bifurcation

It follows from Theorem 2.2 that diffusion and delay can induce Turing-Hopf bifurcation under certain conditions. In this section, we choose d_2 and τ as the perturbation parameters of Turing-Hopf bifurcation and always assume that

(C_1) Eq. (2.3) has one zero root $\lambda_1 = 0$ and a pair of purely imaginary roots $\pm i\omega_{n_c}$ at $(\tau, d_2) = \left(\tau_c, d_2^c\right)$ for some $n_c \in \mathbb{N}$ and $n_c \neq 1$, and all other eigenvalues have negative real parts.

We first calculate the normal form of the Turing-Hopf bifurcation and then investigate the dynamical classification near the Turing-Hopf bifurcation point (d_2^c, τ_c). The calculation of the normal form is based on the results in [14,27,28], where there is no delay in the diffusion and no nonlocal terms, and is further generalized to the single population model with memory-based diffusion and nonlocal effect.

Translating u_* to the origin by setting $\breve{u}(x,t) = u(x,t) - u_*$, normalizing the delay by the time-scaling $t \to t/\tau$, and then dropping the breve for simplification of notation, the first equation of (1.3) becomes

$$\begin{aligned}\frac{\partial u}{\partial t} &= \tau\left(d_1 u_{xx} + d_2 u_* u_{xx}(x,t-1) + d_2 u u_{xx}(x,t-1) + d_2 u_x u_x(x,t-1)\right)\\ &\quad + \tau f(u+u_*, \widehat{u}+u_*).\end{aligned} \tag{3.1}$$

Define the real-valued Sobolev space

$$\mathscr{X} = \left\{ u(x) : u(x) \in W^{2,2}(0,\ell\pi),\ \frac{\partial u(x)}{\partial x}\bigg|_{x=0,\ell\pi} = 0 \right\},$$

and let $\mathscr{C} := C\left([-1,0]; \mathscr{X}\right)$ be the Banach space of continuous mappings from $[-1,0]$ to $\mathscr{X}$.

In what follows, setting $\tau = \tau_c + \mu_1$, $d_2 = d_2^c + \mu_2$ such that $(\mu_1,\mu_2) = (0,0)$ is the Turing-Hopf bifurcation point, (3.1) is written as the following functional differential equation in $\mathscr{C} := C\left([-1,0]; \mathscr{X}\right)$

$$\frac{\partial u}{\partial t} = d(\mu)\Delta(u_t) + L(\mu)(u_t) + F(u_t,\mu), \tag{3.2}$$

where, for simplification of notations, we use $u(t)$ for $u(x,t)$, $u_t \in \mathscr{C}$ for $u_t(\theta) = u(x, t+\theta)$, $-1 \le \theta \le 0$, and for $\varphi \in \mathscr{C}$, $d(\mu)\Delta, L(\mu) : \mathscr{C} \to \mathscr{X}$, $F : \mathscr{C} \times \mathbb{R} \to \mathscr{X}$ are given, respectively, by

$$d(\mu)\Delta(\varphi) = d_0\Delta(\varphi) + F^d(\varphi,\mu),\ \ L(\mu)(\varphi) = (\tau_c + \mu_1)\left(A\varphi(0) + B(\widehat{\varphi}(0))\right),$$

and

$$F(\varphi,\mu) = (\tau_c+\mu_1) f\left(\varphi(0) + u_*, \widehat{\varphi}(0) + u_*\right) - L(\mu)(\varphi), \tag{3.3}$$

where

$$\begin{aligned} d_0\Delta(\varphi) &= \tau_c d_1 \varphi_{xx}(0) + \tau_c d_2^c u_* \varphi_{xx}(-1),\\ F^d(\varphi,\mu) &= \frac{1}{2}F_2^d(\varphi,\mu) + \frac{1}{3!}F_3^d(\varphi,\mu) + \frac{1}{4!}F_4^d(\varphi,\mu)\end{aligned} \tag{3.4}$$

with

$$\begin{aligned} F_2^d(\varphi,\mu) &= 2d_1\mu_1\varphi_{xx}(0) + 2d_2^c u_*\mu_1\varphi_{xx}(-1) + 2\tau_c u_*\mu_2\varphi_{xx}(-1)\\ &\quad + 2\tau_c d_2^c \varphi_x(0)\varphi_x(-1) + 2\tau_c d_2^c \varphi(0)\varphi_{xx}(-1),\end{aligned} \tag{3.5}$$

$$\begin{aligned} F_3^d(\varphi,\mu) &= 6u_*\mu_1\mu_2\varphi_{xx}(-1) + 6d_2^c\mu_1\varphi_x(0)\varphi_x(-1) + 6\tau_c\mu_2\varphi_x(0)\varphi_x(-1),\\ &\quad + 6d_2^c\mu_1\varphi(0)\varphi_{xx}(-1) + 6\tau_c\mu_2\varphi(0)\varphi_{xx}(-1),\\ F_4^d(\varphi,\mu) &= 24\mu_1\mu_2\varphi_x(0)\varphi_x(-1) + 24\mu_1\mu_2\varphi(0)\varphi_{xx}(-1).\end{aligned} \tag{3.6}$$

Since the perturbation parameters μ_1 and μ_2 are treated as variables in the calculation of normal forms, separating the linear terms from the nonlinear terms, (3.2) can be written as

$$\frac{\partial u}{\partial t} = d_0\Delta(u_t) + L_0(u_t) + \widetilde{F}(u_t, \mu_1, \mu_2), \tag{3.7}$$

where $L_0(\varphi) = \tau_c\left(A\varphi(0) + B\widehat{\varphi}(0)\right)$ and

$$\widetilde{F}(\varphi, \mu_1, \mu_2) = \mu_1\left(A\varphi(0) + B\widehat{\varphi}(0)\right)) + F(\varphi, \mu) + F^d(\varphi, \mu). \tag{3.8}$$

Let

$$\mathcal{M}_0(\lambda) = \lambda - \tau_c(A + B),\ \ \mathcal{M}_n(\lambda) = \lambda - \tau_c\left(-\left(d_1 + d_2u_*e^{-\lambda}\right)\left(\frac{n}{\ell}\right)^2 + A\right),\ n \in \mathbb{N},$$

then the characteristic equation for the linearized system

$$\frac{\partial u}{\partial t} = d_0\Delta(u_t) + L_0(u_t) \tag{3.9}$$

is

$$\prod_{n\in\mathbb{N}_0}\mathcal{M}_n(\lambda) = 0. \tag{3.10}$$

Comparing (2.3) with (3.10), we know that under the condition (C_1), Eq. (3.10) has one zero root $\lambda_1 = 0$ and a pair of purely imaginary roots $\pm i\omega_c$ at $(d_2, \tau) = (d_2^c, \tau_c)$, and all other eigenvalues have negative real parts, where

$$\omega_c = \tau_c\omega_{n_c}.$$

Notice that the eigenvalue problem (2.2) has eigenvalues $\mu_n = (n/\ell)^2$, with the associated normalized eigenfunctions

$$\gamma_n(x) = \frac{\cos\left(\frac{nx}{\ell}\right)}{\|\cos\left(\frac{nx}{\ell}\right)\|_{2,2}} = \begin{cases} \frac{1}{\sqrt{\ell\pi}}, & \text{for } n = 0, \\ \frac{\sqrt{2}}{\sqrt{\ell\pi}}\cos\left(\frac{nx}{\ell}\right), & \text{for } n \neq 0, \end{cases}$$

where the norm $\|\cdot\|_{2,2}$ is induced by the inner product $[\cdot,\cdot]$ as follows

$$[u, v] = \int_0^{\ell\pi} uv\,dx,\ \text{for } u, v \in \mathscr{X}.$$

It follows from this inner product that

$$\left[\gamma_i(x), \gamma_j(x)\right] = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{3.11}$$

Let $\mathscr{B}_n = \text{span}\left\{\left[v(\cdot), \gamma_n(x)\right]\gamma_n(x) |\ v \in \mathscr{C}\right\}$. Then it is easy to verify that

$$L_0(\mathscr{B}_n) \subset \text{span}\left\{\gamma_n(x)\right\},\ n \in \mathbb{N}_0.$$

Assume that $z_t(\theta) \in C = C([-1,0],\mathbb{R})$ and $z_t(\theta)\gamma_n(x) \in \mathscr{B}_n$. Then, on $\mathscr{B}_n$, the linear equation (3.9) is equivalent to the following functional differential equation (FDE) in $C := C([-1,0],\mathbb{R})$

$$\dot{z}(t) = \mathscr{L}_n\left(z_t(\theta)\right), \tag{3.12}$$

where

$$\mathscr{L}_n\left(z_t(\theta)\right) = L_n^d\left(z_t(\theta)\right) + L_0^n(z_t(\theta)) \tag{3.13}$$

with

$$L_n^d\left(z_t(\theta)\right) = -\tau_c(n/\ell)^2\left(d_1 z_t(0) + d_2^c u_* z_t(-1)\right),\ L_0^n(z_t(\theta)) = \begin{cases} \tau_c\left(A+B\right)z_t(0), & n=0, \\ \tau_c A z_t(0), & n \in \mathbb{N}. \end{cases}$$

The characteristic equation of linear system (3.12) is the same as given in (3.10).

Define $\eta_n(\theta) \in BV([-1,0],\mathbb{R})$ such that

$$\int_{-1}^{0} d\eta_n(\theta)\varphi(\theta) = L_n^d(\varphi(\theta)) + L_0(\varphi(\theta)),\quad \varphi \in C,$$

and use the adjoint bilinear form on $C^* \times C$, $C^* = C([0,1],\mathbb{R}^*)$, as follows

$$\langle\psi(s),\varphi(\theta)\rangle_n = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)d\eta_n(\theta)\varphi(\xi)d\xi,\quad \text{for } \psi \in C^*, \varphi \in C.$$

Let $\Lambda = \{i\omega_c, -i\omega_c, 0\}$. Denote the generalized eigenspace of (3.12) associated with Λ by P_n and the corresponding adjoint space by $P_n^*, n = n_c, 1$. Then, by the adjoint theory of functional differential equation [18], C can be decomposed as $C = P_n \oplus Q_n$, $n = 1, n_c$, where $Q_n = \{\varphi \in C : \langle\psi,\varphi\rangle = 0, \forall\psi \in P_n^*\}$. Choose the bases $\Phi_n(\theta)$ and $\Psi_n(s)$ of P_n and P_n^*, respectively, as follows

$$\Phi_1(\theta) = 1,\ \Phi_{n_c}(\theta) = \left(e^{i\omega_c\theta},\ e^{-i\omega_c\theta}\right),\ \Psi_1(s) = \psi_1,\ \Psi_{n_c}(s) = \left(\psi_{n_c}e^{-i\omega_c s},\ \overline{\psi_{n_c}}e^{i\omega_c s}\right)^T$$

such that $\langle\Psi_{n_c}(s),\Phi_{n_c}(\theta)\rangle_{n_c} = I_2$ (the 2×2 identity matrix) and $\langle\Psi_1(s),\Phi_1(\theta)\rangle_1 = 1$, where $\overline{\psi_{n_c}}$ is the complex conjugate of ψ_{n_c}. These two identifies are verified if and only if

$$\psi_1 = \frac{1}{1-\tau_c(1/\ell)^2 d_2^c u_*},\quad \psi_{n_c} = \frac{1}{1-\tau_c(n_c/\ell)^2 d_2^c u_* e^{-i\omega_c}}. \tag{3.14}$$

Following [14] and [28], we define $\mathscr{C}_0^1 = \left\{\phi \in \mathscr{C} : \dot{\phi} \in \mathscr{C}, \phi(0) \in \mathrm{dom}(d\Delta)\right\}$ and let

$$\Phi(\theta) = \begin{pmatrix} \Phi_{n_c}(\theta) & \Phi_1(\theta) \end{pmatrix}, z_x = \left(z_1(t)\gamma_{n_c}(x), z_2(t)\gamma_{n_c}(x), z_3(t)\gamma_1(x)\right)^T.$$

For $\varphi_t(x,\theta) \in \mathscr{C}_0^1$, we have the following decomposition

$$\varphi_t(x,\theta) = \Phi(\theta)z_x + w, \quad w \in \mathscr{C}_0^1 \cap \mathrm{Ker}\pi := \mathscr{Q}^1. \tag{3.15}$$

Then, system (3.7) is decomposed as a system of abstract ODEs on $\mathbb{R}^3 \times \mathrm{Ker}\pi$

$$\begin{cases} \dot{z} = Bz + \Psi(0)\begin{pmatrix} \left[\widetilde{F}\left(\Phi(\theta)z_x + w, \mu\right), \gamma_{n_c}(x)\right] \\ \left[\widetilde{F}\left(\Phi(\theta)z_x + w, \mu\right), \gamma_1(x)\right] \end{pmatrix}, \\ \dot{w} = A_{\mathscr{Q}^1} w + (I - \pi)X_0(\theta)\widetilde{F}\left(\Phi(\theta)z_x + w, \mu\right), \end{cases} \tag{3.16}$$

where $A_{\mathscr{Q}^1} : \mathscr{Q}^1 \to \mathrm{Ker}\pi$, and $B = \mathrm{diag}\{i\omega_c, -i\omega_c, 0\}$, $\Psi(0) = \mathrm{diag}\left\{\Psi_{n_c}(0), \Psi_1(0)\right\}$.

Consider the formal Taylor expansion

$$\widetilde{F}(\varphi,\mu) = \frac{1}{2}\widetilde{F}_2(\varphi,\mu) + \frac{1}{3!}\widetilde{F}_3(\varphi,\mu) + \cdots. \tag{3.17}$$

Then (3.16) is formally written as

$$\dot{z} = Bz + \sum_{j\geq 2} \tfrac{1}{j!} f_j^1(z, w, \mu),$$
$$\tfrac{d}{dt} w = A_{\mathscr{Q}^1} w + \sum_{j\geq 2} \tfrac{1}{j!} f_j^2(z, w, \mu),$$

where

$$f_j^1(z,w,\mu) = \Psi(0)\begin{pmatrix} \left[\widetilde{F}_j\left(\Phi(\theta)z_x + w, \mu\right), \gamma_{n_c}(x)\right] \\ \left[\widetilde{F}_j\left(\Phi(\theta)z_x + w, \mu\right), \gamma_1(x)\right] \end{pmatrix}, \tag{3.18}$$

$$f_j^2(z,w,\mu) = (I - \pi)X_0(\theta)\widetilde{F}_j\left(\Phi(\theta)z_x + w, \mu\right). \tag{3.19}$$

In terms of the normal form theory of partial functional differential equations [14], after a recursive transformation of variables of the form

$$(z, w) = (\widetilde{z}, \widetilde{w}) + \frac{1}{j!}\left(U_j^1(\widetilde{z},\mu), U_j^2(\widetilde{z},\mu)\right), j \geq 2,$$

where $z, \widetilde{z} \in \mathbb{R}^3$, $w, \widetilde{w} \in \mathscr{Q}^1$ and $U_j^1 : \mathbb{R}^5 \to \mathbb{R}^3$, $U_j^2 : \mathbb{R}^5 \to \mathscr{Q}^1$ are homogeneous polynomials of degree j in $\widetilde{z}$ and α, one can obtain the following normal form

$$\dot{z} = Bz + \sum_{j\geq 2} \frac{1}{j!} g_j^1(z, 0, \mu), \tag{3.20}$$

which is the normal form as in the usual sense for ODEs.

In the following, we calculate

$$g_2^1(z,0,\mu) = \mathrm{Proj}_{\mathrm{Ker}(M_2^1)} f_2^1(z,0,\mu),$$

and

$$g_3^1(z,0,0) = \mathrm{Proj}_{S_1} \widetilde{f}_3^1(z,0,0), \tag{3.21}$$

where $\widetilde{f}_3^1(z,0,\mu)$ is the terms of order 3 in (z,μ) obtained after performing the change of variables of order 2, and $\mathrm{Ker}\left(M_2^1\right)$ and S (see [28]) are as follows

$$\begin{aligned}\mathrm{Ker}(M_2^1) = \mathrm{span}&\left\{\begin{pmatrix} z_1z_3\\0\\0\end{pmatrix},\begin{pmatrix} z_1\mu_i\\0\\0\end{pmatrix},\begin{pmatrix} 0\\z_2z_3\\0\end{pmatrix},\begin{pmatrix} 0\\z_2\mu_i\\0\end{pmatrix},\right.\\ &\left.\begin{pmatrix} 0\\0\\z_1z_2\end{pmatrix},\begin{pmatrix} 0\\0\\z_3^2\end{pmatrix},\begin{pmatrix} 0\\0\\z_3\mu_i\end{pmatrix},\begin{pmatrix} 0\\0\\\mu_1\mu_2\end{pmatrix},\begin{pmatrix} 0\\0\\\mu_i^2\end{pmatrix}\right\},\end{aligned} \tag{3.22}$$

$$S = \mathrm{span}\left\{\begin{pmatrix} z_1^2z_2\\0\\0\end{pmatrix},\begin{pmatrix} z_1z_3^2\\0\\0\end{pmatrix},\begin{pmatrix} 0\\z_1z_2^2\\0\end{pmatrix},\begin{pmatrix} 0\\z_2z_3^2\\0\end{pmatrix},\begin{pmatrix} 0\\0\\z_1z_2z_3\end{pmatrix},\begin{pmatrix} 0\\0\\z_3^3\end{pmatrix}\right\}. \tag{3.23}$$

The Taylor expansion of $f(u+u_*,\widehat{u}+u_*)$ at $(u,\widehat{u})=(0,0)$ can be written as

$$f(u+u_*,\widehat{u}+u_*) = Au + B\widehat{u} + \sum_{i+j\geq 2}\frac{1}{i!j!}f_{ij}u^i\widehat{u}^j,$$

where

$$f_{ij} = \left.\frac{\partial^{i+j} f(u+u_*,\widehat{u}+u_*)}{\partial u^i\partial\widehat{u}^j}\right|_{(u,\widehat{u})=(0,0)}.$$

Then, for $\varphi = \Phi(\theta)z_x + w, \in \mathscr{C}_0^1$, from (3.3) and (3.8), we have

$$\widetilde{F}(\varphi,\mu) = A\mu_1\varphi(0) + B\mu_1\widehat{\varphi}(0) + (\tau_c+\mu_1)\sum_{i+j\geq 2}\frac{1}{i!j!}f_{ij}\varphi^i(0)\widehat{\varphi}^j(0) + F^d(\varphi(\theta),\mu), \tag{3.24}$$

where $F^d(\varphi(\theta),\mu)$ is the nonlinear term induced by the memory-based diffusion and is defined by (3.5).

So,

$$\frac{1}{2!}\widetilde{F}_2(\varphi,\mu) = A\mu_1\varphi(0) + B\mu_1\widehat{\varphi}(0) + \frac{1}{2!}F_2(\varphi,\mu) + \frac{1}{2!}F_2^d(\varphi(\theta),\mu), \tag{3.25}$$

$$\frac{1}{3!}\widetilde{F}_3(\varphi,\mu) = \frac{1}{3!}F_3(\varphi,\mu) + \frac{1}{3!}F_3^d(\varphi(\theta),\mu), \tag{3.26}$$

where

$$F_2(\varphi,\mu)=2\tau_c\sum_{i+j=2}\frac{1}{i!j!}f_{ij}\varphi^i(0)\widehat{\varphi}^j(0) \tag{3.27}$$

and

$$F_3(\varphi,\mu)=6\mu_1\sum_{i+j=2}\frac{1}{i!j!}f_{ij}\varphi^i(0)\widehat{\varphi}^j(0)+6\tau_c\sum_{i+j=3}\frac{1}{i!j!}f_{ij}\varphi^i(0)\widehat{\varphi}^j(0). \tag{3.28}$$

In the following subsections, we introduce the following notation for simplification

$$\mathcal{H}\left(\alpha z_1^{q_1}z_2^{q_2}z_3^{q_3}\mu_1^{\iota_1}\mu_2^{\iota_2}\right)=\begin{pmatrix}\alpha z_1^{q_1}z_2^{q_2}z_3^{q_3}\mu_1^{\iota_1}\mu_2^{\iota_2}\\ \overline{\alpha}z_1^{q_2}z_2^{q_1}z_3^{q_3}\mu_1^{\iota_1}\mu_2^{\iota_2}\end{pmatrix},\ \alpha\in\mathbb{C}.$$

3.1. Calculation of $g_2^1(z,0,\mu)$

Notice that

$$\Phi(\theta)z_x=z_1(t)e^{i\omega_c\theta}\gamma_{n_c}(x)+z_2(t)e^{-i\omega_c\theta}\gamma_{n_c}(x)+z_3(t)\gamma_1(x),\quad \widehat{\Phi(\theta)z_x}=0. \tag{3.29}$$

By (3.27), we have $F_2(\Phi(\theta)z_x,\mu)=F_2(\Phi(\theta)z_x,0)$. Then, we let

$$F_2(\Phi(\theta)z_x,0)=\sum_{q_1+q_2+q_3=2}A_{q_1q_2q_3}\gamma_{n_c}^{q_1+q_2}(x)\gamma_1^{q_3}(x)z_1^{q_1}z_2^{q_2}z_3^{q_3},$$

where

$$A_{200}=A_{020}=A_{002}=\tau_c f_{20},\ A_{110}=A_{101}=A_{011}=2\tau_c f_{20}.$$

From (3.5), we write

$$\begin{aligned}&F_2^d(\Phi(\theta)z_x,0)\\ =&\sum_{q_1+q_2+q_3=2}A_{q_1q_2q_3}^{(d,1)}\left((n_c/\ell)\xi_{n_c}(x)\right)^{q_1+q_2}\left((1/\ell)\xi_1(x)\right)^{q_3}z_1^{q_1}z_2^{q_2}z_3^{q_3}\\ &-(n_c/\ell)^2A_{200}^{(d,2)}\gamma_{n_c}^2(x)z_1^2-(n_c/\ell)^2A_{020}^{(d,2)}\gamma_{n_c}^2(x)z_2^2-(1/\ell)^2A_{002}^{(d,2)}\gamma_1^2(x)z_3^2\\ &-(n_c/\ell)^2A_{110}^{(d,2)}\gamma_{n_c}^2(x)z_1z_2-\gamma_{n_c}(x)\gamma_1(x)\left(\left((n_c/\ell)^2A_{101}^{(d,2)}+(1/\ell)^2A_{101}^{(d,3)}\right)z_1z_3\right.\\ &-\gamma_{n_c}(x)\gamma_1(x)\left((n_c/\ell)^2A_{011}^{(d,2)}+(1/\ell)^2A_{011}^{(d,3)}\right)z_2z_3,\end{aligned} \tag{3.30}$$

where

$$\xi_n(x)=\frac{\sin\left(\frac{nx}{\ell}\right)}{\|\cos\left(\frac{nx}{\ell}\right)\|_{2,2}}=\begin{cases}0, & \text{for } n=0,\\ \frac{\sqrt{2}}{\sqrt{\ell\pi}}\sin\left(\frac{nx}{\ell}\right), & \text{for } n\neq 0,\end{cases}$$

and

$$\begin{aligned}
&A_{200}^{(d,1)}=2\tau_c d_2^c e^{-i\omega_c}=\overline{A_{020}^{(d,1)}}, && A_{002}^{(d,1)}=2\tau_c d_2^c,\\
&A_{110}^{(d,1)}=4\tau_c d_2^c \mathrm{Re}\left\{e^{i\omega_c}\right\}, && A_{101}^{(d,1)}=2\tau_c d_2^c\left(1+e^{-i\omega_c}\right)=\overline{A_{011}^{(d,1)}},\\
&A_{200}^{(d,2)}=2\tau_c d_2^c e^{-i\omega_c}=\overline{A_{020}^{(d,2)}}, && A_{002}^{(d,2)}=2\tau_c d_2^c,\quad A_{110}^{(d,2)}=4\tau_c d_2^c \mathrm{Re}\left\{e^{i\omega_c}\right\},\\
&A_{101}^{(d,2)}=2\tau_c d_2^c e^{-i\omega_c}=\overline{A_{011}^{(d,2)}}, && A_{101}^{(d,3)}=2\tau_c d_2^c=A_{011}^{(d,3)}.
\end{aligned}$$

It is easy to verify that

$$\int_0^{\ell\pi}\gamma_{n_c}^3(x)dx=\int_0^{\ell\pi}\gamma_{n_c}^2(x)\gamma_1(x)dx=\int_0^{\ell\pi}\gamma_1^3(x)dx=0,$$

$$\int_0^{\ell\pi}\xi_{n_c}^2(x)\gamma_{n_c}(x)dx=\int_0^{\ell\pi}\xi_{n_c}(x)\xi_1(x)\gamma_{n_c}(x)dx=\int_0^{\ell\pi}\xi_{n_c}^2(x)\gamma_1(x)dx=0,$$

$$\int_0^{\ell\pi}\xi_1^2(x)\gamma_1(x)dx=0,\quad \int_0^{\ell\pi}\xi_{n_c}(x)\xi_1(x)\gamma_1(x)dx=\begin{cases}\frac{1}{\sqrt{2\ell\pi}}, & n_c=2,\\ 0, & n_c\neq 2.\end{cases}$$

$$\int_0^{\ell\pi}\gamma_1^2(x)\gamma_{n_c}(x)dx=\begin{cases}\frac{1}{\sqrt{2\ell\pi}}, & n_c=2,\\ 0, & n_c\neq 2,\end{cases}\quad \int_0^{\ell\pi}\xi_1^2(x)\gamma_{n_c}(x)dx=\begin{cases}-\frac{1}{\sqrt{2\ell\pi}}, & n_c=2,\\ 0, & n_c\neq 2.\end{cases}$$

In addition, we have $\widetilde{F}_2(\Phi(\theta)z_x,0)=F_2(\Phi(\theta)z_x,0)+F_2^d(\Phi(\theta)z_x,0)$. Then, by a direct calculation, we have

$$\begin{aligned}
f_2^1(z,0,0)&=\Psi(0)\begin{pmatrix}\left[\widetilde{F}_2(\Phi(\theta)z_x,0),\gamma_{n_c}(x)\right]\\ \left[\widetilde{F}_2(\Phi(\theta)z_x,0),\gamma_1(x)\right]\end{pmatrix}\\
&=\begin{cases}\frac{1}{\sqrt{2\ell\pi}}\Psi(0)\begin{pmatrix}\widetilde{A}_{002}z_3^2\\ \widetilde{A}_{101}z_1z_3+\widetilde{A}_{011}z_2z_3\end{pmatrix}, & n_c=2,\\ (0\ \ 0)^T, & n_c\neq 2,\end{cases}
\end{aligned}\tag{3.31}$$

where

$$\begin{aligned}
\widetilde{A}_{002}&=A_{002}-(1/\ell)^2A_{002}^{(d,1)}-(1/\ell)^2A_{002}^{(d,2)},\\
\widetilde{A}_{101}&=A_{101}+(n_c/\ell^2)A_{101}^{(d,1)}-\left((n_c/\ell)^2A_{101}^{(d,2)}+(1/\ell)^2A_{101}^{(d,3)}\right),\\
\widetilde{A}_{011}&=A_{011}+(n_c/\ell^2)A_{011}^{(d,1)}-\left((n_c/\ell)^2A_{011}^{(d,2)}+(1/\ell)^2A_{011}^{(d,3)}\right).
\end{aligned}\tag{3.32}$$

Clearly, $\mathrm{Proj}_{\mathrm{Ker}(M_2^1)}f_2^1(z,0,0)=(0\ \ 0)^T$. Therefore, it is easy from (3.11), (3.18), (3.22), (3.25) and (3.29) to verify that

$$\begin{aligned}\tfrac{1}{2}g_2^1(z,0,\mu) &= \tfrac{1}{2}\mathrm{Proj}_{\mathrm{Ker}(M_2^1)} f_2^1(z,0,\mu)\\ &= \begin{pmatrix}\mathcal{H}\left((B_{11}\mu_1+B_{21}\mu_2)z_1\right)\\ (B_{13}\mu_1+B_{23}\mu_2)\,z_3\end{pmatrix},\end{aligned} \tag{3.33}$$

where

$$\begin{aligned} &B_{11}=i\omega_{n_c}\psi_{n_c}, && B_{21}=-\psi_{n_c}(n_c/\ell)^2\tau_c u_* e^{-i\omega_c},\\ &B_{13}=\psi_1\left(A-(1/\ell)^2\left(d_1+d_2^c u_*\right)\right)=0, && B_{23}=-\psi_1(1/\ell)^2\tau_c u_*.\end{aligned} \tag{3.34}$$

3.2. Calculation of $g_3^1(z,0,0)$

Denote

$$f_2^{(1,1)}(z,w,0)=\Psi(0)\begin{pmatrix}\left[F_2\left(\Phi(\theta)z_x+w,0\right),\gamma_{n_c}(x)\right]\\ \left[F_2\left(\Phi(\theta)z_x+w,0\right),\gamma_1(x)\right]\end{pmatrix}, \tag{3.35}$$

$$f_2^{(1,2)}(z,w,0)=\Psi(0)\begin{pmatrix}\left[F_2^d\left(\Phi(\theta)z_x+w,0\right),\gamma_{n_c}(x)\right]\\ \left[F_2^d\left(\Phi(\theta)z_x+w,0\right),\gamma_1(x)\right]\end{pmatrix}. \tag{3.36}$$

In addition, it follows from (3.33) that $g_2^1(z,0,0)=(0\;0)^T$. Then $\widetilde{f}_3^1(z,0,0)$ is determined by

$$\begin{aligned}&\widetilde{f}_3^1(z,0,0)\\ =\;& f_3^1(z,0,0)+\tfrac{3}{2}\Big[\left(D_z f_2^1(z,0,0)\right)U_2^1(z,0)+\left(D_{w,\widehat{w}} f_2^{(1,1)}(z,0,0)\right)\widetilde{U}_2^2(z,0)(\theta)\\ &+\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right)U_2^{(2,d)}(z,0)(\theta)\Big],\end{aligned} \tag{3.37}$$

where $f_2^1(z,0,0)=f_2^{(1,1)}(z,0,0)+f_2^{(1,2)}(z,0,0)$,

$$U_2^1(z,0)=\left(M_2^1\right)^{-1}\mathrm{Proj}_{Im(M_2^1)} f_2^1(z,0,0),\quad U_2^2(z,0)(\theta)=\left(M_2^2\right)^{-1} f_2^2(z,0,0),$$

and

$$\widetilde{U}_2^{(2,d)}(z,0)(\theta)=\left(U_2^{(2,d)}(z,0)(\theta)\widehat{U}_2^{(2,d)}(z,0)(\theta)\right)^T,$$

$$U_2^{(2,d)}(z,0)(\theta)=\left(U_2^2(z,0)(\theta),U_{2x}^2(z,0)(\theta),U_{2xx}^2(z,0)(\theta)\right)^T.$$

Next, we compute $\mathrm{Proj}_S\widetilde{f}_3^1(z,0,0)$ step by step according to (3.37). The calculation is divided into the following four subsections.

3.2.1. The calculation of $\mathrm{Proj}_S f_3^1(z,0,0)$

From (3.6), we have $F_3^d(\Phi(\theta)z_x,0)=0$, which, together with (3.26) and (3.29), yields

$$\widetilde{F}_3(\Phi(\theta)z_x,0)=F_3(\Phi(\theta)z_x,0)=\sum_{q_1+q_2+q_3=3} A_{q_1q_2q_3}\gamma_{n_c}^{q_1+q_2}(x)\gamma_1^{q_3}(x)z_1^{q_1}z_2^{q_2}z_3^{q_3},\, q_1,q_2,q_3\in\mathbb{N}_0,$$

where

$$A_{300}=A_{030}=A_{003}=\tau_c f_{30},\, A_{111}=6\tau_c f_{30},$$
$$A_{210}=A_{201}=A_{120}=A_{102}=A_{021}=A_{012}=3\tau_c f_{30}.$$

Then, by (3.18), we have

$$f_3^1(z,0,0)=\Psi(0)\begin{pmatrix}\sum\limits_{q_1+q_2+q_3=3} A_{q_1q_2q_3}\int_0^{\ell\pi}\gamma_{n_c}^{q_1+q_2+1}(x)\gamma_1^{q_3}(x)dx z_1^{q_1}z_2^{q_2}z_3^{q_3}\\ \sum\limits_{q_1+q_2+q_3=3} A_{q_1q_2q_3}\int_0^{\ell\pi}\gamma_{n_c}^{q_1+q_2}(x)\gamma_1^{q_3+1}(x)dx z_1^{q_1}z_2^{q_2}z_3^{q_3}\end{pmatrix},$$

which, together with the fact that

$$\int_0^{\ell\pi}\gamma_{n_c}^4(x)dx=\int_0^{\ell\pi}\gamma_1^4(x)dx=\frac{3}{2\ell\pi},\quad \int_0^{\ell\pi}\gamma_{n_c}^2(x)\gamma_1^2(x)dx=\frac{1}{\ell\pi},$$

leads to

$$\frac{1}{3!}\mathrm{Proj}_S f_3^1(z,0,0)=\begin{pmatrix}\mathcal{H}\left(C_{210}z_1^2z_2+C_{102}z_1z_3^2\right)\\ C_{111}z_1z_2z_3+C_{003}z_3^3\end{pmatrix}, \tag{3.38}$$

where

$$C_{210}=\frac{1}{4\ell\pi}\psi_{n_c}A_{210},\; C_{102}=\frac{1}{6\ell\pi}\psi_{n_c}A_{102},\; C_{111}=\frac{1}{6\ell\pi}\psi_1A_{111},\; C_{003}=\frac{1}{4\ell\pi}\psi_1A_{003}. \tag{3.39}$$

3.2.2. The calculation of $\mathrm{Proj}_S\left(D_zf_2^1\right)(z,0,0)U_2^1(z,0)$

By (3.31), we obtain that for $n_c\neq 2$, $U_2^1(z,0)=(0,0)^T$, and for $n_c=2$,

$$\begin{aligned}&U_2^1(z,0)\\ &=\left(M_2^1\right)^{-1}\mathrm{Proj}_{\mathrm{Im}M_2^1}f_2^1(z,0,0)\\ &=\frac{1}{i\omega_c\sqrt{2\ell\pi}}\begin{pmatrix}-\psi_{n_c}\widetilde{A}_{002}z_3^2\\ \overline{\psi_{n_c}}\widetilde{A}_{002}z_3^2\\ \psi_1\left(\widetilde{A}_{101}z_1z_3-\widetilde{A}_{011}z_2z_3\right)\end{pmatrix}.\end{aligned}$$

So,

$$\frac{1}{3!}\mathrm{Proj}_S\left[\left(D_z f_2^1\right)(z,0,0)U_2^1(z,0)\right]=\begin{pmatrix}\mathcal{H}\left(D_{210}z_1^2z_2+D_{102}z_1z_3^2\right)\\ D_{111}z_1z_2z_3+D_{003}z_3^3\end{pmatrix}, \tag{3.40}$$

where $D_{210}=D_{111}=0$, and

$$D_{102}=\begin{cases}\frac{\psi_{n_c}\widetilde{A}_{002}\psi_1\widetilde{A}_{101}}{6\ell\pi\omega_c i}, & n_c=2,\\ 0, & n_c\neq 2,\end{cases}\quad D_{003}=\begin{cases}-\frac{\mathrm{Im}\{\psi_1\widetilde{A}_{101}\psi_{n_c}\widetilde{A}_{002}\}}{6\ell\pi\omega_c}, & n_c=2,\\ 0, & n_c\neq 2.\end{cases}$$

3.2.3. The calculation of $\mathit{Proj}_S\left(\left(D_w f_2^{(1,1)}(z,0,0)\right)\left(\widetilde{U}_2^2(z,0)(\theta)\right)\right)$

Let

$$U_2^2(z,0)(\theta)\triangleq h(z,\theta)=\sum_{n\in\mathbb{N}_0}h_n(z,\theta)\gamma_n(x),$$

where

$$h_n(z,\theta)=\sum_{q_1+q_2+q_3=2}h_{n,q_1q_2q_3}(\theta)z_1^{q_1}z_2^{q_2}z_3^{q_3}.$$

Then, we have

$$h_x(z,\theta)=-\sum_{n\in\mathbb{N}_0}(n/\ell)h_n(z,\theta)\xi_n(x),\quad h_{xx}(\theta,z)=-\sum_{n\in\mathbb{N}_0}(n/\ell)^2h_n(z,\theta)\gamma_n(x),$$

and

$$\widehat{U_2^2}(z,0)(\theta)\triangleq\widehat{h}(z,\theta)=\frac{1}{\sqrt{\ell\pi}}h_0(z,\theta).$$

From (3.27), $F_2(\Phi(\theta)z_x+w,\mu)$ can be written as

$$\begin{aligned}&F_2(\Phi(\theta)z_x+w,\mu)=F_2(\Phi(\theta)z_x+w,0)\\ &=\sum_{q_1+q_2+q_3=2}A_{q_1q_2q_3}\gamma_{n_c}^{q_1+q_2}(x)\gamma_1^{q_3}(x)z_1^{q_1}z_2^{q_2}z_3^{q_3}\\ &\quad+\mathcal{S}_2\left(\Phi(\theta)z_x,w\right)+\widehat{\mathcal{S}}_2\left(\Phi(\theta)z_x,\widehat{w}\right)+O\left(|(w,\widehat{w})|^2\right),\end{aligned} \tag{3.41}$$

where

$$\begin{aligned}\mathcal{S}_2\left(\Phi(\theta)z_x,w\right)&=2\tau_c f_{20}\left(z_1(t)\gamma_{n_c}(x)+z_2(t)\gamma_{n_c}(x)+z_3(t)\gamma_1(x),0\right)w(0),\\ \widehat{\mathcal{S}}_2\left(\Phi(\theta)z_x,\widehat{w}\right)&=2\tau_c f_{11}\left(z_1(t)\gamma_{n_c}(x)+z_2(t)\gamma_{n_c}(x)+z_3(t)\gamma_1(x),0\right)\widehat{w}(0).\end{aligned}$$

By a direct calculation, we have

$$\int_0^{\ell\pi} \gamma_k(x)\gamma_n(x)\gamma_k(x)dx = \begin{cases} \frac{1}{\sqrt{\ell\pi}}, & n=0, \\ \frac{1}{\sqrt{2\ell\pi}}, & n=2k, \qquad k=1,\ n_c. \\ 0, & n\neq 0, 2k, \end{cases} \tag{3.42}$$

$$\begin{aligned} \int_0^{\ell\pi} \gamma_1(x)\gamma_n(x)\gamma_{n_c}(x)dx &= \int_0^{\ell\pi} \gamma_{n_c}(x)\gamma_n(x)\gamma_1(x)dx \\ &= \begin{cases} \frac{1}{\sqrt{2\ell\pi}}, & n=n_c-1, \\ \frac{1}{\sqrt{2\ell\pi}}, & n=n_c+1, \\ 0, & n\neq n_c-1, n_c+1. \end{cases} \end{aligned} \tag{3.43}$$

Denote $\widetilde{U}_2^2(z,0)(\theta) = \left(h(z,\theta), \widehat{h}(z,\theta)\right)^T$. Then, by (3.35) and (3.41), we have

$$\frac{1}{3!}\mathrm{Proj}_S\left(D_{w,\widehat{w}} f_2^{(1,1)}(z,0,0)\left(\widetilde{U}_2^2(z,0)(\theta)\right)\right) = \begin{pmatrix} \mathcal{H}\left(E_{210}z_1^2z_2 + E_{102}z_1z_3^2\right) \\ E_{111}z_1z_2z_3 + E_{003}z_3^3 \end{pmatrix}, \tag{3.44}$$

where

$$\begin{aligned} E_{210} &= \tfrac{1}{3}\tau_c\psi_{n_c}\left\{\tfrac{f_{20}}{\sqrt{\ell\pi}}\left(h_{0,110}(0)+h_{0,200}(0)\right) + \tfrac{f_{20}}{\sqrt{2\ell\pi}}\left(h_{2n_c,110}(0)+h_{2n_c,200}(0)\right)\right. \\ &\quad \left. + \tfrac{f_{11}}{\sqrt{\ell\pi}}\left(h_{0,110}(0)+h_{0,200}(0)\right)\right\}, \\ E_{102} &= \tfrac{1}{3}\tau_c\psi_{n_c}\left\{\tfrac{f_{20}}{\sqrt{\ell\pi}}h_{0,002}(0) + \tfrac{f_{20}}{\sqrt{2\ell\pi}}h_{2n_c,002}(0)\right. \\ &\quad \left. + \tfrac{f_{20}}{\sqrt{2\ell\pi}}\left(h_{n_c-1,101}(0)+h_{n_c+1,101}(0)\right) + \tfrac{f_{11}}{\sqrt{\ell\pi}}h_{0,002}(0)\right\}, \\ E_{111} &= \tfrac{1}{6}\tau_c\psi_1\left\{\tfrac{f_{20}}{\sqrt{\ell\pi}}h_{0,110}(0) + \tfrac{f_{11}}{\sqrt{\ell\pi}}h_{0,110}(0) + \tfrac{f_{20}}{\sqrt{2\ell\pi}}h_{2,110}(0)\right. \\ &\quad \left. + \tfrac{f_{20}}{\sqrt{2\ell\pi}}\left(h_{n_c-1,011}(0)+h_{n_c-1,101}(0)+h_{n_c+1,011}(0)+h_{n_c+1,101}(0)\right)\right\}, \\ E_{003} &= \tfrac{1}{3}\tau_c\psi_1\left\{\tfrac{f_{20}}{\sqrt{\ell\pi}}h_{0,002}(0) + \tfrac{f_{11}}{\sqrt{\ell\pi}}h_{0,002}(0) + \tfrac{f_{20}}{\sqrt{2\ell\pi}}h_{2,002}(0)\right\}. \end{aligned} \tag{3.45}$$

3.2.4. *The calculation of Proj$_S\left(\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right) U_2^{(2,d)}(z,0)(\theta)\right)$*

Letting

$$\phi(\theta) = = \Phi(\theta)z_x = \left(z_1e^{i\omega_c\theta} + z_2e^{-i\omega_c\theta}\right)\gamma_{n_c}(x) + z_3\gamma_1(x),$$

we have

$$\phi_x(\theta) = -(n_c/\ell)\left(z_1e^{i\omega_c\theta} + z_2e^{-i\omega_c\theta}\right)\xi_{n_c}(x) - (1/\ell)z_3\xi_1(x),$$

and

$$\phi_{xx}(\theta) = -(n_c/\ell)^2\left(z_1e^{i\omega_c\theta} + z_2e^{-i\omega_c\theta}\right)\gamma_{n_c}(x) - (1/\ell)^2z_3\gamma_1(x).$$

Denote

$$
\begin{aligned}
&F_2^d(\phi(\theta), w, w_x, w_{xx}) = F_2^d(\phi(\theta)+w, \mu) \\
= \ & 2d_1\mu_1\left(\phi_{xx}(0)+w_{xx}(0)\right) + 2d_2^c u_*\mu_1\left(\phi_{xx}(-1)+w_{xx}(-1)\right) \\
&+2\tau_c u_*\mu_2\left(\phi_{xx}(-1)+w_{xx}(-1)\right) + 2\tau_c d_2^c\left(\phi_x(0)+w_x(0)\right)\left(\phi_x(-1)+w_x(-1)\right) \\
&+2\tau_c d_2^c\left(\phi(0)+w(0)\right)\left(\phi_{xx}(-1)+w_{xx}(-1)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\widetilde{S}_2^{(d,1)}(\phi(\theta), w) = 2\tau_c d_2^c \phi_{xx}(-1) w(0), \\
&\widetilde{S}_2^{(d,2)}(\phi(\theta), w_x) = 2\tau_c d_2^c\left(\phi_x(-1)w_x(0)+\phi_x(0)w_x(-1)\right), \\
&\widetilde{S}_2^{(d,3)}(\phi(\theta), w_{xx}) = 2\tau_c d_2^c \phi(0) w_{xx}(-1).
\end{aligned}
\tag{3.46}
$$

Then, we have

$$
\begin{aligned}
&D_{w,w_x,w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx})\big|_{w,w_x,w_{xx}=0} U_2^{(2,d)}(z,0)(\theta) \\
= \ & \widetilde{S}_2^{(d,1)}\left(\phi(\theta), h(\theta,z)\right) + \widetilde{S}_2^{(d,2)}\left(\phi(\theta), h_x(\theta,z)\right) \\
&+\widetilde{S}_2^{(d,3)}\left(\phi(\theta), h_{xx}(\theta,z)\right).
\end{aligned}
\tag{3.47}
$$

In addition, it is easy to calculate that

$$
\int_0^{\ell\pi} \xi_k(x)\xi_n(x)\gamma_k(x)dx = \begin{cases} \frac{1}{\sqrt{2\ell\pi}}, & n=2k, \\ 0, & n\neq 2k, \end{cases} \quad k=1,n_c,
\tag{3.48}
$$

$$
\int_0^{\ell\pi} \xi_1(x)\xi_n(x)\gamma_{n_c}(x)dx = \begin{cases} -\frac{1}{\sqrt{2\ell\pi}}, & n=n_c-1, \\ \frac{1}{\sqrt{2\ell\pi}}, & n=n_c+1, \\ 0, & n\neq n_c-1, n_c+1, \end{cases}
\tag{3.49}
$$

$$
\int_0^{\ell\pi} \xi_{n_c}(x)\xi_n(x)\gamma_1(x)dx = \begin{cases} \frac{1}{\sqrt{2\ell\pi}}, & n=n_c-1, \\ \frac{1}{\sqrt{2\ell\pi}}, & n=n_c+1, \\ 0, & n\neq n_c-1, n_c+1. \end{cases}
\tag{3.50}
$$

Therefore, by (3.42), (3.43), (3.48), (3.49), (3.50) and a direct calculation, we have

$$
\begin{aligned}
&\left[\widetilde{S}_2^{(d,1)}\left(\phi(\theta), h(\theta,z)\right), \gamma_k(x)\right] \\
= &\begin{cases} -2\tau_c d_2^c\Big((n_c/\ell)^2(z_1e^{-i\omega_c}+z_2e^{i\omega_c})\left(\frac{1}{\sqrt{\ell\pi}}h_0(0,z)+\frac{1}{\sqrt{2\ell\pi}}h_{2n_c}(0,z)\right) \\ +(1/\ell)^2\frac{1}{\sqrt{2\ell\pi}}z_3\left(h_{n_c-1}(0,z)+h_{n_c+1}(0,z)\right)\Big), \quad k=n_c, \\ -2\tau_c d_2^c\Big((n_c/\ell)^2\frac{1}{\sqrt{2\ell\pi}}(z_1e^{-i\omega_c}+z_2e^{i\omega_c})\left(h_{n_c-1}(0,z)+h_{n_c+1}(0,z)\right) \\ +(1/\ell)^2 z_3\left(\frac{1}{\sqrt{\ell\pi}}h_0(0,z)+\frac{1}{\sqrt{2\ell\pi}}h_2(0,z)\right)\Big), \quad k=1, \end{cases}
\end{aligned}
$$

$$\left[\widetilde{S}_2^{(d,2)}\left(\phi(\theta), h_x(\theta,z)\right), \gamma_k(x)\right]$$

$$= \begin{cases} \frac{2\tau_c d_2^c}{\sqrt{2\ell\pi}}\left((2n_c^2/\ell^2)\left((z_1 e^{-i\omega_c}+z_2 e^{i\omega_c})h_{2n_c}(0,z)+(z_1+z_2)h_{2n_c}(-1,z)\right)\right. \\ +(1/\ell^2)\left(z_3\left((n_c+1)h_{n_c+1}(0,z)-(n_c-1)h_{n_c-1}(0,z)\right)\right. \\ \left.+(1/\ell^2)z_3\left((n_c+1)h_{n_c+1}(-1,z)-(n_c-1)h_{n_c-1}(-1,z)\right)\right), \quad k=n_c, \\ \frac{2\tau_c d_2^c}{\sqrt{2\ell\pi}}\left((n_c/\ell^2)\left((z_1 e^{-i\omega_c}+z_2 e^{i\omega_c})\left((n_c-1)h_{n_c-1}(0,z)+(n_c+1)h_{n_c+1}(0,z)\right)\right)\right. \\ +(n_c/\ell^2)(z_1+z_2)\left((n_c-1)h_{n_c-1}(-1,z)+(n_c+1)h_{n_c+1}(-1,z)\right) \\ \left.+(2/\ell^2)\left(z_3\left(h_2(0,z)+h_2(-1,z)\right)\right)\right), \quad k=1, \end{cases}$$

and

$$\left[\widetilde{S}_2^{(d,3)}\left(\phi(\theta), h_{xx}(\theta,z)\right), \gamma_k(x)\right]$$

$$= \begin{cases} -\frac{2\tau_c d_2^c}{\sqrt{2\ell\pi}}\left((2n_c/\ell)^2(z_1+z_2)h_{2n_c}(-1,z)\right. \\ \left.+z_3\left(((n_c-1)/\ell)^2 h_{n_c-1}(-1,z)+((n_c+1)/\ell)^2 h_{n_c+1}(-1,z)\right)\right), \quad k=n_c, \\ -\frac{2\tau_c d_2^c}{\sqrt{2\ell\pi}}\left((z_1+z_2)\left(((n_c-1)/\ell)^2 h_{n_c-1}(-1,z)+((n_c+1)/\ell)^2 h_{n_c+1}(-1,z)\right)\right. \\ \left.+(2/\ell)^2 z_3 h_2(-1,z)\right), \quad k=1. \end{cases}$$

Then, from (3.36), (3.46), (3.47), we have

$$\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right) U_2^{(2,d)}(z,0)(\theta)$$

$$= \Psi(0)\begin{pmatrix} \left[D_{w,w_x,w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx})\big|_{w,w_x,w_{xx}=0} U_2^{(2,d)}(z,0)(\theta),\ \gamma_{n_c}(x)\right] \\ \left[D_{w,w_x,w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx})\big|_{w,w_x,w_{xx}=0} U_2^{(2,d)}(z,0)(\theta),\ \gamma_1(x)\right] \end{pmatrix}$$

and then we obtain

$$\frac{1}{3!}\mathrm{Proj}_S\left(\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right) U_2^{(2,d)}(z,0)(\theta)\right) = \begin{pmatrix} \mathcal{H}\left(E_{210}^d z_1^2 z_2 + E_{102}^d z_1 z_3^2\right) \\ E_{111}^d z_1 z_2 z_3 + E_{003}^d z_3^3 \end{pmatrix}, \tag{3.51}$$

where

$$
\begin{cases}
E^d_{210} = -\frac{1}{3}\psi_{n_c}\tau_c d_2^c (n_c/\ell)^2 e^{-i\omega_c}\left(\frac{1}{\sqrt{\ell\pi}}h_{0,110}(0)+\frac{1}{\sqrt{2\ell\pi}}h_{2n_c,110}(0)\right)\\
\quad -\frac{1}{3}\psi_{n_c}\tau_c d_2^c (n_c/\ell)^2 e^{i\omega_c}\left(\frac{1}{\sqrt{\ell\pi}}h_{0,200}(0)+\frac{1}{\sqrt{2\ell\pi}}h_{2n_c,200}(0)\right)\\
\quad +\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2n_c^2/\ell^2)\left(e^{-i\omega_c}h_{2n_c,110}(0)+h_{2n_c,110}(-1)\right)\\
\quad +\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2n_c^2/\ell^2)\left(e^{i\omega_c}h_{2n_c,200}(0)+h_{2n_c,200}(-1)\right)\\
\quad -\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2n_c/\ell)^2\left(h_{2n_c,110}(-1)+h_{2n_c,200}(-1)\right),\\
E^d_{102} = -\frac{1}{3}\psi_{n_c}\tau_c d_2^c (n_c/\ell)^2 e^{-i\omega_c}\left(\frac{1}{\sqrt{\ell\pi}}h_{0,002}(0)+\frac{1}{\sqrt{2\ell\pi}}h_{2n_c,002}(0)\right)\\
\quad -\frac{1}{3\sqrt{2\ell\pi}}\psi_{n_c}\tau_c d_2^c (1/\ell)^2\left(h_{n_c-1,101}(0)+h_{n_c+1,101}(0)\right)\\
\quad +\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2n_c^2/\ell^2)\left(e^{-i\omega_c}h_{2n_c,002}(0)+h_{2n_c,002}(-1)\right)\\
\quad +\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(1/\ell^2)(n_c+1)\left(h_{n_c+1,101}(0)+h_{n_c+1,101}(-1)\right)\\
\quad -\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(1/\ell^2)(n_c-1)\left(h_{n_c-1,101}(0)+h_{n_c-1,101}(-1)\right)\\
\quad -\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2n_c/\ell)^2 h_{2n_c,002}(-1)\\
\quad -\frac{\psi_{n_c}\tau_c d_2^c}{3\sqrt{2\ell\pi}}(1/\ell^2)\left((n_c-1)^2h_{n_c-1,101}(-1)+(n_c+1)^2h_{n_c+1,101}(-1)\right),
\end{cases}
\tag{3.52}
$$

and

$$
\begin{cases}
E^d_{111} = -\frac{1}{3\sqrt{2\ell\pi}}\psi_1\tau_c d_2^c (n_c/\ell)^2 e^{-i\omega_c}\left(h_{n_c-1,011}(0)+h_{n_c+1,011}(0)\right)\\
\quad -\frac{1}{3\sqrt{2\ell\pi}}\psi_1\tau_c d_2^c (n_c/\ell)^2 e^{i\omega_c}\left(h_{n_c-1,101}(0)+h_{n_c+1,101}(0)\right)\\
\quad -\frac{1}{3}\psi_1\tau_c d_2^c (1/\ell)^2\left(\frac{1}{\sqrt{\ell\pi}}h_{0,110}(0)+\frac{1}{\sqrt{2\ell\pi}}h_{2,110}(0)\right)\\
\quad +\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(n_c/\ell^2)e^{-i\omega_c}\left((n_c-1)h_{n_c-1,011}(0)+(n_c+1)h_{n_c+1,011}(0)\right)\\
\quad +\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(n_c/\ell^2)e^{i\omega_c}\left((n_c-1)h_{n_c-1,101}(0)+(n_c+1)h_{n_c+1,101}(0)\right)\\
\quad +\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(n_c/\ell^2)\left((n_c-1)h_{n_c-1,011}(-1)+(n_c+1)h_{n_c+1,011}(-1)\right)\\
\quad +\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(n_c/\ell^2)\left((n_c-1)h_{n_c-1,101}(-1)+(n_c+1)h_{n_c+1,101}(-1)\right)\\
\quad +\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2/\ell^2)\left(h_{2,110}(0)+h_{2,110}(-1)\right)\\
\quad -\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}\left(((n_c-1)/\ell)^2h_{n_c-1,011}(-1)+((n_c+1)/\ell)^2h_{n_c+1,011}(-1)\right)\\
\quad -\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}\left(((n_c-1)/\ell)^2h_{n_c-1,101}(-1)+((n_c+1)/\ell)^2h_{n_c+1,101}(-1)\right)\\
\quad -\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2/\ell)^2h_{2,110}(-1),\\
E^d_{003} = -\frac{1}{3}\psi_1\tau_c d_2^c (1/\ell)^2\left(\frac{1}{\sqrt{\ell\pi}}h_{0,002}(0)+\frac{1}{\sqrt{2\ell\pi}}h_{2,002}(0)\right)\\
\quad +\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2/\ell^2)\left(h_{2,002}(0)+h_{2,002}(-1)\right)\\
\quad -\frac{\psi_1\tau_c d_2^c}{3\sqrt{2\ell\pi}}(2/\ell)^2h_{2,002}(-1).
\end{cases}
\tag{3.53}
$$

Clearly, we still need to compute $h_{k,200}(\theta)$ with $k=0,2n_c$, $h_{k,002}(\theta)$ with $k=0,2,2n_c$, $h_{k,110}(\theta)$ with $k=0,2,2n_c$, and $h_{k,101}(\theta), h_{k,011}(\theta)$ with $k=n_c-1,n_c+1$. From [14], we have

$$\begin{aligned}&M_2^2\left(h_n(z,\theta)\gamma_n(x)\right)\\ =\ &\left(D_z\left(h_n(z,\theta)\gamma_n(x)\right)Bz\right)-A_{\mathcal{Q}^1}\left(h_n(z,\theta)\gamma_n(x)\right),\end{aligned}$$

where $A_{\mathscr{Q}^1}:\mathscr{Q}^1\to \mathrm{Ker}\pi$ is defined by

$$A_{\mathscr{Q}^1}w=\dot{w}+X_0(\theta)\left(\mathscr{L}_n(w)-\dot{w}(0)\right),$$

with $\mathscr{L}_n(w)$ being defined by (3.13).

Then, we have

$$\begin{aligned}&M_2^2\left(\sum_{n\in\mathbb{N}_0}h_n(z,\theta)\gamma_n(x)\right)\\ =&\sum_{n\in\mathbb{N}_0}\left(i\omega_c\left(2h_{n,200}(\theta)z_1^2+h_{n,101}(\theta)z_1z_3-2h_{n,020}(\theta)z_2^2-h_{n,011}(\theta)z_2z_3\right)\right.\\ &\left.-\left(\dot{h}_n(z,\theta)+X_0(\theta)\left[\mathscr{L}_n\left(h_n(z,\theta)\right)-\dot{h}_n(z,0)\right]\right)\right)\gamma_n(x).\end{aligned}\tag{3.54}$$

$$\left[M_2^2\left(\sum_{n\in\mathbb{N}_0}h_n(z,\theta)\gamma_n(x)\right),\gamma_k(x)\right]=\left[f_2^2(z,0,0),\gamma_k(x)\right].\tag{3.55}$$

By (3.16), we obtain

$$\begin{aligned}f_2^2(z,0,0)&=(I-\pi)X_0(\theta)\widetilde{F}_2\left(\Phi(\theta)z_x,0\right)\\ &=X_0(\theta)\widetilde{F}_2\left(\Phi(\theta)z_x,0\right)-\Phi_{n_c}(\theta)\Psi_{n_c}(0)\left[\widetilde{F}_2\left(\Phi(\theta)z_x,0\right),\gamma_{n_c}(x)\right]\gamma_{n_c}(x)\\ &\quad-\Phi_1(\theta)\Psi_1(0)\left[\widetilde{F}_2\left(\Phi(\theta)z_x,0\right),\gamma_1(x)\right]\gamma_1(x).\end{aligned}$$

So,

$$\left[f_2^2(z,0,0),\gamma_k(x)\right]=\begin{cases}\frac{1}{\sqrt{\ell\pi}}X_0(\theta)\left(A_{200}z_1^2+A_{020}z_2^2+A_{002}z_3^2+A_{110}z_1z_2\right), & k=0,\\ \frac{1}{\sqrt{2\ell\pi}}X_0(\theta)\left(\widetilde{A}_{002}z_3^2\right), & k=2,n_c\neq 3,\\ \frac{1}{\sqrt{2\ell\pi}}X_0(\theta)\left(\widetilde{A}_{002}z_3^2+\widetilde{A}_{101}z_1z_3+\widetilde{A}_{011}z_2z_3\right), & k=2,n_c=3,\\ \frac{1}{\sqrt{2\ell\pi}}X_0(\theta)\left(\widetilde{A}_{200}z_1^2+\widetilde{A}_{020}z_2^2+\widetilde{A}_{110}z_1z_2\right), & k=2n_c,\\ \frac{1}{\sqrt{2\ell\pi}}X_0(\theta)\left(\widetilde{A}_{101}z_1z_3+\widetilde{A}_{011}z_2z_3\right), & k=n_c-1,\\ & n_c\neq 3,\\ \frac{1}{\sqrt{2\ell\pi}}X_0(\theta)\left(\widetilde{A}^-_{101}z_1z_3+\widetilde{A}^-_{011}z_2z_3\right), & k=n_c+1,\end{cases}\tag{3.56}$$

where $\widetilde{A}_{002}$, $\widetilde{A}_{101}$ and $\widetilde{A}_{011}$ are defined by (3.32), and

$$\widetilde{A}_{200} = A_{200} - (n_c/\ell)^2 A_{200}^{(d,1)} - (n_c/\ell)^2 A_{200}^{(d,2)},$$
$$\widetilde{A}_{020} = A_{020} - (n_c/\ell)^2 A_{020}^{(d,1)} - (n_c/\ell)^2 A_{020}^{(d,2)},$$
$$\widetilde{A}_{110} = A_{110} - (n_c/\ell)^2 A_{110}^{(d,1)} - (n_c/\ell)^2 A_{110}^{(d,2)},$$
$$\widetilde{A}_{101}^{-} = A_{101} - (n_c/\ell^2) A_{101}^{(d,1)} - \left((n_c/\ell)^2 A_{101}^{(d,2)} + (1/\ell)^2 A_{101}^{(d,3)}\right),$$
$$\widetilde{A}_{011}^{-} = A_{011} - (n_c/\ell^2) A_{011}^{(d,1)} - \left((n_c/\ell)^2 A_{011}^{(d,2)} + (1/\ell)^2 A_{011}^{(d,3)}\right).$$

Hence, from (3.54), (3.55) and (3.56) and matching the coefficients of z_1z_2, z_1z_3, z_2z_3 and z_1^2, z_3^2, we have

$$h_{0,q_1q_2q_3}(\theta): \begin{cases} z_1^2, & \begin{cases} \dot{h}_{0,200}(\theta) - 2i\omega_c h_{0,200}(\theta) = 0, \\ \dot{h}_{0,200}(0) - \mathscr{L}_0\left(h_{0,200}(\theta)\right) = \frac{1}{\sqrt{\ell\pi}} A_{200}, \end{cases} \\ z_3^2, & \begin{cases} \dot{h}_{0,002}(\theta) = 0, \\ \dot{h}_{0,002}(0) - \mathscr{L}_0\left(h_{0,002}(\theta)\right) = \frac{1}{\sqrt{\ell\pi}} A_{002}, \end{cases} \\ z_1z_2, & \begin{cases} \dot{h}_{0,110}(\theta) = 0, \\ \dot{h}_{0,110}(0) - \mathscr{L}_0\left(h_{0,110}(\theta)\right) = \frac{1}{\sqrt{\ell\pi}} A_{110}, \end{cases} \end{cases} \tag{3.57}$$

$$h_{2,q_1q_2q_3}(\theta), n_c \neq 3: \begin{cases} z_3^2, & \begin{cases} \dot{h}_{2,002}(\theta) = 0, \\ \dot{h}_{2,002}(0) - \mathscr{L}_2\left(h_{2,002}(\theta)\right) = \frac{1}{\sqrt{2\ell\pi}} \widetilde{A}_{002}, \end{cases} \\ z_1z_2, & \begin{cases} \dot{h}_{2,110}(\theta) = 0, \\ \dot{h}_{2,110}(0) - \mathscr{L}_2\left(h_{2,110}(\theta)\right) = 0, \end{cases} \end{cases} \tag{3.58}$$

$$h_{2n_c,q_1q_2q_3}(\theta): \begin{cases} z_1^2, & \begin{cases} \dot{h}_{2n_c,200}(\theta) - 2i\omega_c h_{2n_c,200}(\theta) = 0, \\ \dot{h}_{2n_c,200}(0) - \mathscr{L}_{2n_c}\left(h_{2n_c,200}(\theta)\right) = \frac{1}{\sqrt{2\ell\pi}} \widetilde{A}_{200}, \end{cases} \\ z_3^2, & \begin{cases} \dot{h}_{2n_c,002}(\theta) = 0, \\ \dot{h}_{2n_c,002}(0) - \mathscr{L}_{2n_c}\left(h_{2n_c,002}(\theta)\right) = 0, \end{cases} \\ z_1z_2, & \begin{cases} \dot{h}_{2n_c,110}(\theta) = 0, \\ \dot{h}_{2n_c,110}(0) - \mathscr{L}_{2n_c}\left(h_{2n_c,110}(\theta)\right) = \frac{1}{\sqrt{2\ell\pi}} \widetilde{A}_{110}, \end{cases} \end{cases} \tag{3.59}$$

$$h_{n_c-1,q_1q_2q_3}(\theta): \begin{cases} z_1z_3, & \begin{cases} \dot{h}_{n_c-1,101}(\theta) - i\omega_c h_{n_c-1,101}(\theta) = 0, \\ \dot{h}_{n_c-1,101}(0) - \mathscr{L}_{n_c-1}\left(h_{n_c-1,101}(\theta)\right) = \frac{1}{\sqrt{2\ell\pi}} \widetilde{A}_{101}, \end{cases} \\ z_2z_3, & \begin{cases} \dot{h}_{n_c-1,011}(\theta) + i\omega_c h_{n_c-1,011}(\theta) = 0, \\ \dot{h}_{n_c-1,011}(0) - \mathscr{L}_{n_c-1}\left(h_{n_c-1,011}(\theta)\right) = \frac{1}{\sqrt{2\ell\pi}} \widetilde{A}_{011}, \end{cases} \end{cases} \tag{3.60}$$

$$h_{n_c+1,q_1q_2q_3}(\theta):\begin{cases} z_1z_3, & \begin{cases}\dot{h}_{n_c+1,101}(\theta)-i\omega_c h_{n_c+1,101}(\theta)=0,\\ \dot{h}_{n_c+1,101}(0)-\mathscr{L}_{n_c+1}\left(h_{n_c+1,101}(\theta)\right)=\frac{1}{\sqrt{2\ell\pi}}\widetilde{A}^{-}_{101},\end{cases}\\ z_2z_3, & \begin{cases}\dot{h}_{n_c+1,011}(\theta)+i\omega_c h_{n_c+1,011}(\theta)=0,\\ \dot{h}_{n_c+1,011}(0)-\mathscr{L}_{n_c+1}\left(h_{n_c+1,011}(\theta)\right)=\frac{1}{\sqrt{2\ell\pi}}\widetilde{A}^{-}_{011}.\end{cases}\end{cases} \tag{3.61}$$

Solving (3.57), (3.58), (3.59), (3.60) and (3.61), we have

$$\begin{cases} h_{0,200}(\theta)=\frac{1}{\mathcal{M}_0(2i\omega_c)\sqrt{\ell\pi}}A_{200}e^{2i\omega_c\theta},\\ h_{0,002}(\theta)=\frac{1}{\mathcal{M}_0(0)\sqrt{\ell\pi}}A_{002},\\ h_{0,110}(\theta)=\frac{1}{\mathcal{M}_0(0)\sqrt{\ell\pi}}A_{110}.\end{cases} \tag{3.62}$$

$$\begin{cases} h_{2,002}(\theta)=\frac{1}{\mathcal{M}_2(0)\sqrt{2\ell\pi}}\widetilde{A}_{002},\\ h_{2,110}(\theta)=0,\end{cases} \qquad n_c\neq 3, \tag{3.63}$$

$$\begin{cases} h_{2n_c,200}(\theta)=\frac{1}{\mathcal{M}_{2n_c}(2i\omega_c)\sqrt{2\ell\pi}}\widetilde{A}_{200}e^{2i\omega_c\theta},\\ h_{2n_c,002}(\theta)=0,\\ h_{2n_c,110}(\theta)=\frac{1}{\mathcal{M}_{2n_c}(0)\sqrt{2\ell\pi}}\widetilde{A}_{110},\end{cases} \tag{3.64}$$

$$\begin{cases} h_{n_c-1,101}(\theta)=\frac{1}{\mathcal{M}_{n_c-1}(i\omega_c)\sqrt{2\ell\pi}}\widetilde{A}_{101}e^{i\omega_c\theta},\\ h_{n_c-1,011}(\theta)=\frac{1}{\mathcal{M}_{n_c-1}(-i\omega_c)\sqrt{2\ell\pi}}\widetilde{A}_{011}e^{-i\omega_c\theta},\end{cases} \tag{3.65}$$

$$\begin{cases} h_{n_c+1,101}(\theta)=\frac{1}{\mathcal{M}_{n_c+1}(i\omega_c)\sqrt{2\ell\pi}}\widetilde{A}^{-}_{101}e^{i\omega_c\theta},\\ h_{n_c+1,011}(\theta)=\frac{1}{\mathcal{M}_{n_c+1}(-i\omega_c)\sqrt{2\ell\pi}}\widetilde{A}^{-}_{011}e^{-i\omega_c\theta}.\end{cases} \tag{3.66}$$

Let

$$B_{210}=C_{210}+\frac{3}{2}\left(D_{210}+E_{210}+E^d_{210}\right),\ B_{102}=C_{102}+\frac{3}{2}\left(D_{102}+E_{102}+E^d_{102}\right),$$
$$B_{111}=C_{111}+\frac{3}{2}\left(D_{111}+E_{111}+E^d_{111}\right),\ B_{003}=C_{003}+\frac{3}{2}\left(D_{003}+E_{003}+E^d_{003}\right).$$

Then, by (3.33), (3.38), (3.40), (3.44) and (3.51), and transforming the system (3.20) to the cylindrical coordinates form, we obtain the normal form truncated to the third order terms for the Turing-Hopf bifurcation as follows

$$\begin{cases}\dot{\rho}=\alpha_1(\mu)\rho+\kappa_{11}\rho^3+\kappa_{12}\rho r^2,\\ \dot{r}=\alpha_2(\mu)r+\kappa_{21}\rho^2 r+\kappa_{22}r^3,\end{cases} \tag{3.67}$$

where

$$\alpha_1(\mu) = \mathrm{Re}(B_{11})\,\mu_1 + \mathrm{Re}(B_{21})\,\mu_2,\ \alpha_2(\mu) = \mathrm{Re}(B_{13})\,\mu_1 + \mathrm{Re}(B_{23})\,\mu_2,$$
$$\kappa_{11} = \mathrm{Re}(B_{210}),\ \kappa_{12} = \mathrm{Re}(B_{102}),\ \kappa_{21} = B_{111},\ \kappa_{22} = B_{003}.$$

It follows from (2.3), (3.14) and (3.34) that

$$\mathrm{Re}(B_{11}) = \frac{\omega_c^2}{\left(1+\tau_c\left(d_1(n_c/\ell)^2 - A\right)\right)^2 + \omega_c^2} > 0,\quad \mathrm{Re}(B_{13}) = 0,$$
$$\mathrm{Re}(B_{21}) = \frac{\tau_c\left(1+\tau_c\left(d_1(n_c/\ell)^2 - A\right)\right)\left(d_1(n_c/\ell)^2 - A\right) + \omega_c^2}{\left(1+\tau_c\left(d_1(n_c/\ell)^2 - A\right)\right)^2 + \omega_c^2},$$
$$\mathrm{Re}(B_{23}) = \mathrm{Re}\left(-\psi_1(1/\ell)^2\tau_c u_*\right) = \frac{-(1/\ell)^2\tau_c u_*}{1-\tau_c(1/\ell)^2 d_2^c u_*}.$$

The quantities κ_{ij} can be determined by the coefficients A, B in the linear terms and the coefficients f_{20}, f_{11} and f_{30} in the nonlinear terms.

Remark 3.1. In the expansion of the nonlinear terms, the nonlocal reaction term should be handled independently and the integral of the eigenfunction should be carefully considered in the calculation of the normal form. The nonlocal reaction term mainly lead to the complexity of the calculation (see 3.2.3) of

$$\mathrm{Proj}_S\left(\left(D_w f_2^{(1,1)}(z,0,0)\right)\left(\widetilde{U}_2^2(z,0)(\theta)\right)\right).$$

The nonlinearity of the memory-based diffusion term $d_2\left(uu_x(x,t-\tau)\right)_x$ yields that the calculation of the normal form is different from the known results in [2,27,28], where the diffusion is linear. This nonlinearity leads to the nonlinear terms F_2^d and F_3^d and we have to calculate

$$\mathrm{Proj}_S\left(\left(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z,0,0)\right) U_2^{(2,d)}(z,0)(\theta)\right),$$

which does not appear for the linear diffusion and significantly increase the complexity of the calculation of the normal form (see Section 3.2.4).

4. Example and numerical simulations

We first consider the following growth function introduced by Britton [5,6] for a single biological population

$$f(u,\widehat{u}) = u(1+\alpha u - \beta u^2 - (1+\alpha-\beta)\widehat{u}), \tag{4.1}$$

where the term αu with $\alpha > 0$ represents an advantage in local aggregation, the term $-\beta u^2$ with $\beta > 0$ mitigates against local crowding, and the term $-(1+\alpha-\beta)\widehat{u}$ with $0 < \beta < 1+\alpha$ represents a disadvantage because of the nonlocal depletion of resources.

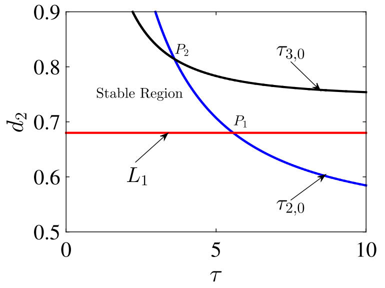

Fig. 2. Stability regions, the steady state bifurcation curve L_1 and Hopf bifurcations $\tau_{2,0}$ and $\tau_{3,0}$ for system (4.1) with $\alpha = 1$, $\beta = 0.3$, $\ell = 2$ and $d_1 = 0.92$. P_1 and P_2 are the Turing-Hopf bifurcation point and the double Hopf bifurcation point, respectively.

For (4.1), we have $u_* = 1$ and $A = \alpha - 2\beta$, $B = -(1 + \alpha - \beta)$.

$$f_{20} = 2(\alpha - 3\beta), \quad f_{11} = -(1 + \alpha - \beta), \quad f_{30} = -6\beta.$$

We are interested in the case of $A > 0$, which is equivalent to $0 < \beta < \alpha/2$. Taking $\alpha = 1$, $\beta = 0.3$, we have $A = 0.4$, $B = -1.7$ and then $A + B < 0$. In the following, we take $\ell = 2$ and focus on two cases: (i) Spatiotemporal dynamics near the Turing-Hopf bifurcation point ($d_1 < A\ell^2$); and (ii) Spatiotemporal dynamics induced by the Hopf and double Hopf bifurcations ($d_1 > A\ell^2$).

4.1. Spatiotemporal dynamics near the Turing-Hopf bifurcation point

For $d_1 = 0.92 < A\ell^2 = 1.6$, the stability region and steady state bifurcation and Hopf bifurcation curves can be illustrated in $\tau - d_2$ plane as shown in Fig. 2 for $0 \le \tau \le 10$ and $0.5 \le d_2 \le 0.9$. In Fig. 2, L_1, defined by $d_2 = -d_1 + A\ell^2 = 0.68$, is the steady state bifurcation curve with $n = 1$; $\tau_{j,0}$, $j = 2, 3$, are Hopf bifurcation curves; the point $P_1(5.5718, 0.68)$ is the Turing-Hopf bifurcation point, which is the intersection point of the steady state bifurcation curve L_1 and the Hopf bifurcation curve $\tau_{2,0}$; and $P_2(3.6146, 0.8140)$ is the double Hopf bifurcation point, which is the intersection point of the Hopf bifurcation curves $\tau_{2,0}$ and $\tau_{3,0}$.

In this subsection, we employ the normal form theory developed in Section 3 to investigate the spatiotemporal dynamics near the Turing-Hopf bifurcation point $P_1(5.5718, 0.68)$. Using the procedure in Section 3 with $n_c = 2$, the normal form truncated to the third order terms is

$$\begin{cases} \dot{\rho} = (0.0506\mu_1 + 1.1996\mu_2)\,\rho - 0.3354\rho^3 + 0.1280\rho r^2, \\ \dot{r} = -26.3725\mu_2 r - 56.4670\rho^2 r - 10.0598 r^3. \end{cases} \tag{4.2}$$

Notice that $\rho > 0$ and r is an arbitrary real number. System (4.2) has a zero equilibrium $A_0(0, 0)$ for any $\mu_1, \mu_2 \in \mathbb{R}$, three boundary equilibria

$$A_1\left(\sqrt{\frac{0.0506\mu_1 + 1.1996\mu_2}{0.3354}},\ 0\right), \quad \mu_2 > -0.0422\mu_1,$$

$$A_2^{\pm}\left(0,\ \pm\sqrt{-\frac{26.3725\mu_2}{10.0598}}\right), \quad \mu_2 < 0,$$

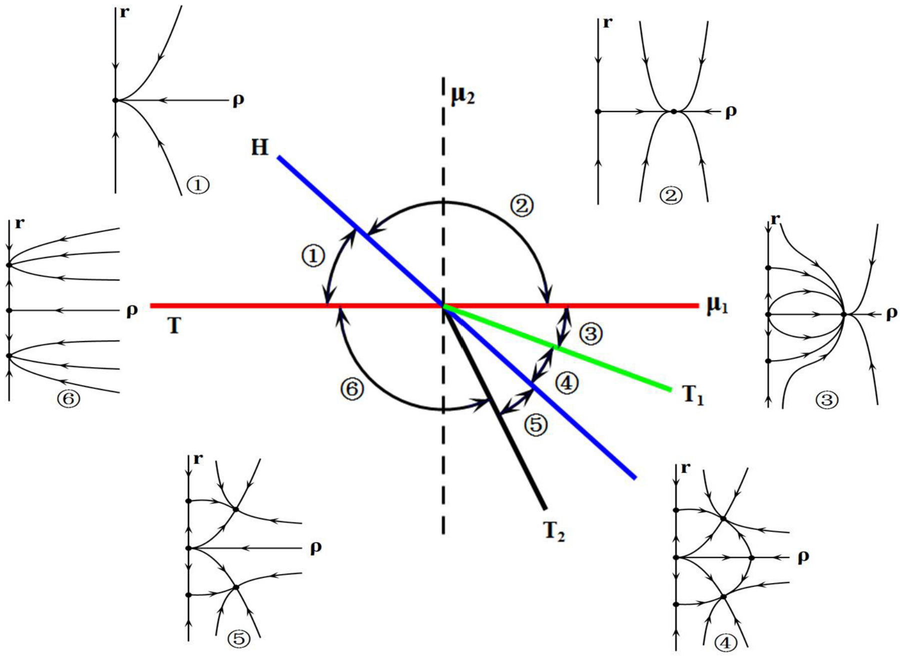

Fig. 3. Bifurcation diagram for the Turing-Hopf bifurcation point P_1.

and two interior equilibria

$$A_3^{\pm}\left(\sqrt{\frac{0.5088\mu_1+8.6930\mu_2}{10.5999}},\ \pm\sqrt{\frac{-2.8562\mu_1-76.5833\mu_2}{10.5999}}\right),$$

for $0.0585\mu_1 < \mu_2 < -0.0373\mu_1$ and $\mu_1 > 0$. Define the critical bifurcation lines as follows:

$$T:\mu_2=0; \quad H:\mu_2=-0.0422\mu_1;$$

$$T_1:\mu_2=-0.0373\mu_1,\ \mu_1>0; \quad T_2:\mu_2=-0.058\mu_1,\ \mu_1>0.$$

In terms of the stability analysis and the results from [19] with minor revisions, the dynamical classification for (4.2) near Turing-Hopf bifurcation point P_1 is illustrated in Fig. 3. When (μ_1,μ_2) are chosen in different regions of Fig. 3, the stable spatiotemporal dynamics of system (1.3) with (4.1) and $\alpha=1$, $\beta=0.3$, $d_1=0.92$, $\ell=2$ are numerically illustrated in Fig. 4.

When (μ_1,μ_2) locates in Region ①, (4.2) has only one stable zero equilibrium, which implies that the original system (1.3) with (4.1) has only one stable constant equilibrium u_*, as shown in Fig. 4(a).

When (μ_1,μ_2) locates in Region ② or Region ③, (4.2) has one stable boundary equilibrium in the horizontal axis, which implies that the original system has one stable nonhomogeneous periodic solution with the spatial mode like $\cos(x)$, as shown in Figs. 4(b) and 4(c) for $(\mu_1,\mu_2)\in$ ② and $(\mu_1,\mu_2)\in$ ③, respectively. When $(\mu_1,\mu_2)\in$ ③, the original system also has two unstable spatially nonhomogeneous steady states.

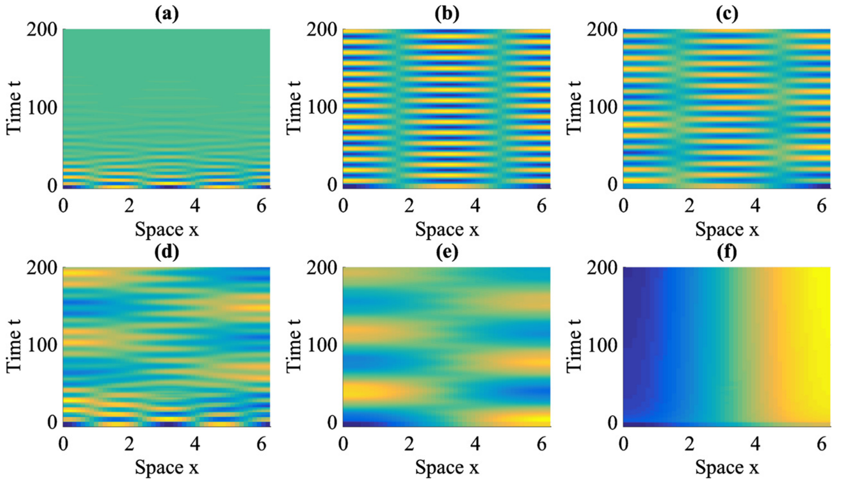

Fig. 4. Spatiotemporal dynamics of system (1.3) with (4.1) and d_1, α, β, ℓ being the same as in Fig. 2, and $\tau = \tau_c + \mu_1, d_2 = d_2^c + \mu_2$, where (μ_1, μ_2) are chosen in different regions of Fig. 2: (a) $(\mu_1, \mu_2) = (1.5718, 0.005) \in$ ①, (b) $(\mu_1, \mu_2) = (0.1718, 0.01) \in$ ②, (c) $(\mu_1, \mu_2) = (0.0992, -0.0002) \in$ ③, (d) $(\mu_1, \mu_2) = (0.4282, -0.015) \in$ ④, (e) $(\mu_1, \mu_2) = (0.1282, -0.007) \in$ ⑤, (f) $(\mu_1, \mu_2) = (-0.5718, -0.08) \in$ ⑥.

When (μ_1, μ_2) locates in Region ④ or Region ⑤, (4.2) has two stable positive equilibria, which implies that the original system has two stable spatially nonhomogeneous periodic solutions with the mixed spatial modes like the combination of $\cos(0.5x)$ and $\cos(x)$, as shown in Figs. 4(d) and 4(e) for $(\mu_1, \mu_2) \in$ ④ and $(\mu_1, \mu_2) \in$ ⑤, respectively. When $(\mu_1, \mu_2) \in$ ④, the original system also has two unstable spatially nonhomogeneous steady states and one unstable spatially nonhomogeneous periodic solution. When $(\mu_1, \mu_2) \in$ ⑤, the original system also has two unstable spatially nonhomogeneous steady states.

When (μ_1, μ_2) locates in Region ⑥, (4.2) has two stable boundary equilibria, which implies that the original system has two stable spatially nonhomogeneous steady states like $\cos(0.5x)$, as shown in Fig. 4(f).

4.2. Spatiotemporal dynamics induced by Hopf and double Hopf bifurcations

For $d_1 = 1.8 > A\ell^2 = 1.6$, the stability region and delay-induced Hopf bifurcation curves can be illustrated in $\tau - d_2$ plane as shown in Fig. 5 for $0 \le \tau \le 10$ and $0 \le d_2 \le 2$. The Hopf bifurcation curves $\tau_{1,0}$ and $\tau_{2,0}$ intersect at $P_3(4.5009, 1.5260)$, and the Hopf bifurcation curves $\tau_{2,0}$ and $\tau_{3,0}$ intersect at $P_4(2.6944, 1.6894)$. P_3 and P_4 are the double Hopf bifurcation points.

According to Theorem 2.2 and direct calculation, we can conclude that

(i) for $d_2 \in [0, 0.2]$, there is no delay-induced Hopf bifurcation and the positive equilibrium $u_* = 1$ is asymptotically stable for any $\tau \ge 0$;
(ii) for $d_2 \in (0.2, 1.4]$, there are Hopf bifurcation values $\tau_{1,j}$ and $\tau_* = \tau_{1,0}$;
(iii) for $d_2 \in (1.4, 1.6222]$, there are Hopf bifurcation values $\tau_{1,j}$, $\tau_{2,j}$ and $\tau_* = \min\{\tau_{1,0}, \tau_{2,0}\}$;
(iv) for $d_2 \in (1.6222, 1.7]$, there are Hopf bifurcation values $\tau_{1,j}$, $\tau_{2,j}$, $\tau_{2,j}$ and

$$\tau_* = \min\{\tau_{1,0}, \tau_{2,0}, \tau_{3,0}\}.$$

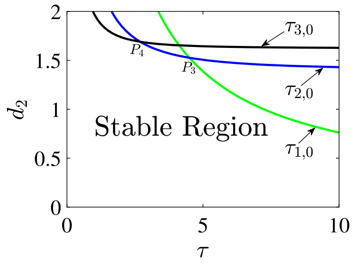

Fig. 5. Stability regions and Hopf bifurcations for system (4.1) where $d_1 = 1.8$ and α, β, ℓ are the same as in Fig. 2.

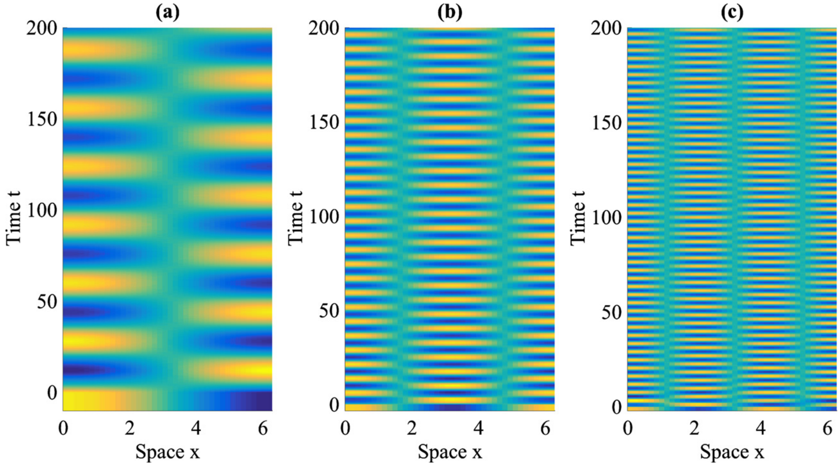

Fig. 6. Delay-induced stable spatially nonhomogeneous periodic solution for different values of d_2 and τ and other parameters as the same in Fig. 5. (a) for $d_2 = 0.8 \in (0.2, 1.5260)$, $\tau = 10 > \tau_{1,0} = 9.4161$; (b) for $d_2 = 1.65 \in (1.5260, 1.6894)$, $\tau = 3.2 > \tau_{2,0} = 2.9591$; (c) for $d_2 = 1.75 \in (1.6894, 2)$, $\tau = 1.9 > \tau_{3,0} = 1.8667$.

Furthermore, it follows from Fig. 5 that

$$\tau_* = \begin{cases} \tau_{1,0}, & d_2 \in (0.2, 1.5260), \\ \tau_{2,0}, & d_2 \in (1.5260, 1.6894), \\ \tau_{3,0}, & d_2 \in (1.6894, 2). \end{cases}$$

Fig. 6(a), Fig. 6(b) and Fig. 6(c) illustrate these Hopf bifurcating periodic solutions at $\tau_{1,0}$, $\tau_{2,0}$ and $\tau_{3,0}$, respectively. Fig. 6(a), Fig. 6(b) and Fig. 6(c) also numerically show the existence of stable spatially nonhomogeneous periodic solutions with the different spatial modes like $\cos(x/2)$, $\cos(x)$ and $\cos(3x/2)$, respectively.

The interaction of Hopf bifurcations can lead to complex dynamics. To study the dynamical classification near the double Hopf bifurcation points P_3 and P_4, one should calculate the normal forms of the double Hopf bifurcations. Since the calculation of the normal forms of the double Hopf bifurcations is complicated, we omit it here and only made the numerical simulations. Figs. 7(b) and 7(d) illustrates the spatiotemporal dynamics due to the interaction of Hopf bifurcations when (τ, d_2) is close to P_3 and P_4, respectively.

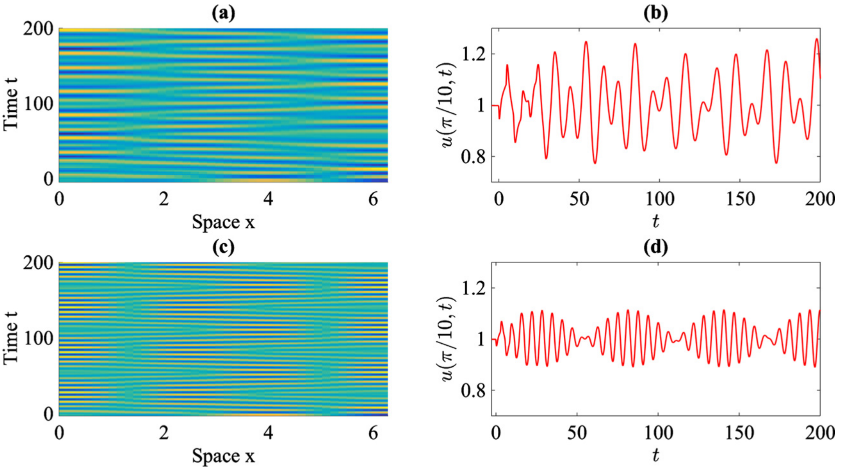

Fig. 7. (a) and (b) are the projection of the surface $u(x,t)$ in the $x-t$ plane, showing the spatiotemporal dynamics due to the interaction of Hopf bifurcations. (b) and (d) are the transversal curves of $u(x,t)$ for fixed space variable $x=\pi/10$, showing the dynamics in the time direction. (a)-(b) $(\tau,d_2)=(4.5,1.55)$ is close to the point P_3, (c)-(d) $(\tau,d_2)=(2.7,1.69)$ is close to the point P_4, and other parameters are the same as in Fig. 5.

Figs. 7(b) exhibits an almost-periodic solution and Fig. 7(d) shows an interesting solution like a "beats" which is usually generated through oscillations of two frequencies which are close to each other and the ratio of two frequencies is large. In fact, for the double Hopf bifurcation point $P_4(2.6944, 1.6894)$, it follows from (2.15) that the two frequencies of the two modes corresponding to $\cos(x)$ and $\cos(3x/2)$ are close to $\omega_2=0.9456$ and $\omega_3=1.0612$, respectively, and $K=\frac{\omega_3+\omega_2}{\omega_3-\omega_2}=17.3462$. Thus, the solution like a "beats" as shown in Fig. 7(d) appears.

5. Discussion

In this paper, we introduce the nonlocal reaction term into the single-species spatial memory model originally proposed by Shi et al. [24] and investigate the spatiotemporal dynamics. We find that nonlocal effect and spatial memory diffusion can lead to codimension-two Turing-Hopf bifurcation and double Hopf bifurcation, which can not occur in the classical scalar reaction-diffusion equation.

As far as the local stability of the positive equilibrium u_* is concerned, Shi et al. [24] has shown that the local stability of the positive equilibrium u_* completely depends on the ratio d_2u_*/d_1 but is independent of the time delay. When the nonlocal term is introduced, the stability of the positive equilibrium u_* depends not only on the ratio d_2u_*/d_1 but also on the time delay. The investigation of the stability of the positive equilibrium is distinguished according to whether the coefficient A of the term without nonlocal term in the linearized equation is larger than zero or not. When $A>0$, there exists a stability switch value τ_* such that for some region in the plane of the Fickian diffusion coefficient d_1 and the memory-based diffusion coefficient d_2, the positive equilibrium is stable for $\tau<\tau_*$ and unstable for $\tau>\tau_*$ and double Hopf and Turing-Hopf bifurcations occur, while when $A\leq 0$, there exists no such stability switch value.

For the Hopf bifurcation, in [24], delay-induced Hopf bifurcations exist and meanwhile for any $\tau>0$, there are infinitely many pairs of complex eigenvalues with positive real parts. Thus, there exists no stability switch value which separates stability/instability regimes. However, for the nonlocal system, the stability switch value τ_* exists, which leads to the occurrence of stable spatially nonhomogeneous periodic solutions.

In addition, there is no steady state bifurcation for the model in [24]. However, because of the existence of the nonlocal term, different kinds of bifurcations like steady state bifurcation and Hopf bifurcation occur and their interactions (Turing-Hopf bifurcation and double Hopf bifurcation) yield more complicated dynamics.

The normal form theory in the literature cannot directly apply to a differential equation with memory-based diffusion and nonlocal effect. Especially, there exists no algorithm of Turing-Hopf bifurcation for this case. In this paper, we generalize the normal form theory for a partial differential equation with delays in the absence of delayed diffusion and nonlocal reaction [14] to the general single population model with memory-based diffusion and nonlocal effect, and derive the algorithm for calculating the normal form of Turing-Hopf bifurcation. This normal form theory for Turing-Hopf bifurcation provides a general method to further study a differential equation with memory-based diffusion and nonlocality. This algorithm is also applicable to a system of equations with memory-based diffusion and nonlocality.

Finally, we apply the obtained theoretical results to a single biological population proposed by Britton [5,6]. When the Fickian diffusion is smaller than some critical value, the memory-based diffusion in combination with the time delay can lead to Turing-Hopf and double Hopf bifurcations. The stability region and these bifurcation curves can be completely determined in the $\tau - d_2$ plane. For the Turing-Hopf bifurcation, the analysis of normal form allows clarifying the dynamical classification near this bifurcation point. The neighborhood of the Turing-Hopf bifurcation point is divided into six different regions, each of which has its special dynamics. We find four types of stable solutions: (i) constant equilibrium, (ii) spatially nonhomogeneous periodic solution with pure mode, (iii) spatially nonhomogeneous periodic solution with mixed modes, and (iv) spatially nonhomogeneous steady state; and three types of transition solutions: (i) the solution from spatially nonhomogeneous steady state to spatially nonhomogeneous periodic solution with pure mode, (ii) the solution from spatially nonhomogeneous steady state to spatially nonhomogeneous periodic solution with mixed modes, and (iii) the solution from spatially nonhomogeneous periodic solution with pure mode to the one with mixed modes. We also find either the coexistence of two stable spatially nonhomogeneous steady states or the coexistence of two stable spatially nonhomogeneous periodic solutions near the Turing-Hopf bifurcation. However, when the Fickian diffusion is larger than the critical value, there are Hopf bifurcation and double Hopf bifurcation but no Turing-Hopf bifurcation. For this case, when the memory-based diffusion (d_2) lies on different intervals, we have different stable spatially nonhomogeneous periodic solutions with different modes, and the interaction of these Hopf bifurcations (double Hopf bifurcations) can lead to the occurrence of new types of solutions like almost-periodic solutions or quasi-periodic solutions.

Although the analysis of this paper is restricted to the one-dimensional spatial domain $(0, \ell\pi)$, the method can be easily generalized to the bounded domain Ω in $\mathbb{R}^N$ $(N > 1)$ with the revision of the characteristic value and the corresponding characteristic function of the Laplace operator in high dimensional space.

The following extensions of this paper are intriguing: (i) since the effects of diffusion and time delay are not independent of each other, a spatiotemporal delay or a distributed delay instead of the discrete delay in model (1.3) should be more realistic; (ii) the extension of the single population model to multispecies interaction models with memory-based diffusion and nonlocal effect is more scientifically interesting but more mathematically challenging.

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