Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 347 (2023) 170–211

Journal of Differential Equations

www.elsevier.com/locate/jde

Bifurcations in a diffusive resource-consumer model with distributed memory

Hao Shen [a], Yongli Song [a,*], Hao Wang [b]

[a] *Department of Mathematics, Hangzhou Normal University, Hangzhou, 311121, China*
[b] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada*

Received 10 April 2022; revised 12 October 2022; accepted 24 November 2022

Abstract

Spatial memory is significant in modeling animal movement. For a diffusive consumer-resource model, a memory-based diffusion of consumer can result in richer and more realistic dynamics. In fact, memory-based diffusion is related to the resource distributions in past times because the memory decays over time. We originally propose a consumer-resource model with distributed memory, and then investigate the influence of the weak memory kernel on the stability of the positive constant steady state. When the memory-based diffusion coefficient is negative, the mean delay does not affect the stability of the positive constant steady state; however, when the memory-based diffusion coefficient is positive, the mean delay can lead to the spatially inhomogeneous periodic oscillation patterns. The direction and stability of Turing bifurcation induced by the memory-based diffusion coefficient are calculated by using the methods of Crandall and Rabinowitz, and the direction and stability of Hopf bifurcation induced by the mean delay are determined by the normal form theory.

MSC: 34K18; 92B05; 35B32

Keywords: Resource-consumer model; Spatial memory; Distributed delay; Stability; Turing bifurcation; Hopf bifurcation

* Corresponding author.
E-mail address: songyl@hznu.edu.cn (Y. Song).

https://doi.org/10.1016/j.jde.2022.11.044
0022-0396/

1. Introduction

Spatial memory is the memory of a living creature's spatial locations in the landscape, and many scholars have introduced implicit spatial memory to characterize the movement of animals [10,13,14]. Fagan et al. [5] proposed that spatial memory/cognition of animals (or creatures) is one of the important factors that determine their movement tendency. The influence of the information gained via past visits and environmental information on the animal movement has been recently investigated by Schlägel and Lewis [16]. Based on the assumption that the memory-based diffusion flux is proportional to the population density at present time and the spatial gradient at a particular past time, Shi et al. [20] proposed a single species spatial memory model to describe the influence of the memory on the animal movement. Song et al. [23] extended the model to consider interacting populations and proposed a consumer-resource model with the explicit consumer's spatial memory on past resource distributions. This memory-based diffusion related to the gradient of past resource distributions is called the memory-based diffusion with discrete delay in what follows. Recently, there has been an increasing activity and interest on the study of subjects on memory-based diffusion with discrete delay (see, e.g., [11,18,19,24] and references therein). More general description and summary of existing PDE (partial differential equation) studies can be found in a recent synthesis paper on the PDE guidance for cognitive animal movement [27].

In fact, the information through the last visit to locations is less available for later retrieval as time passes since the memory decays with time. Therefore, from a biological point of view, the gradient-tracking movement based on distributed memory is more realistic than that based on the memory at a particular time before the present time because the temporal distributed delay can reflect better the influence of the consumer's memory on the resource at all times before the current moment on its diffusion. It is well known that the introduction of delays usually destabilizes the system [12], and there are numerous studies devoted to the population dynamical systems with discrete or/and distributed delays [2,3]. It has been shown that the distributed delay is more stable in essence than the discrete delay [3]. The influence of the distributed delay on the dynamics of the scalar memory-based diffusion equation has also been recently investigated in [1,21,25]. However, to the best of our knowledge, there are few researches on the memory-based diffusion with distributed delay.

In this paper, we replace the discrete delay appearing in the memory-based diffusion of consumer-resource model with explicit spatial memory in [23] and formulate a resource-consumer model with distributed memory. Denote $u(x,t), v(x,t)$ by the population densities in the location x at time t of resource and consumer, respectively, and Ω is an open connected subset of $\mathbb{R}^N (N \geq 1)$ with C^2 boundary $\partial\Omega$, which is the common bounded habitat of both species. Then we propose the following model subject to Neuman boundary condition:

$$\begin{cases} u_t(x,t) = d_{11}\Delta u(x,t) + f(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ v_t(x,t) = d_{22}\Delta v(x,t) - d_{21} div(v(x,t)\nabla w(x,t)) + g(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ \dfrac{\partial u(x,t)}{\partial \boldsymbol{n}} = \dfrac{\partial v(x,t)}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $\boldsymbol{n}$ is the unit outer normal vector of the boundary $\partial\Omega$ and

$$w(x,t) = F * u = \int_{-\infty}^{t} F(t-\xi)u(x,\xi)d\xi = \int_{0}^{\infty} F(s)u(x,t-s)ds. \tag{1.2}$$

Here the parameters $d_{11}, d_{22} > 0$ and $d_{21} \in \mathbb{R}$ are the random diffusion coefficient of $u(x,t)$, $v(x,t)$ and the memory-based diffusion coefficient of $v(x,t)$, respectively. For general available resources, we consider d_{21} to be positive, however, when resources are poisonous, the memory-based diffusion coefficient d_{21} becomes negative for some poisonous plants. The functions f and g describe the biological birth/death of resource and consumer, respectively. $F(\cdot)$ is the reasonable kernel function describing the decay of memory with time. Without loss of generality, the kernel function $F(\cdot)$ is assumed to be positive and normalized to unity such that the constant steady state of system (1.1) is the same to the corresponding ODE (ordinary differential equations), i.e.,

$$F(s) \geq 0, \quad \int_{0}^{\infty} F(s)ds = 1.$$

If $F(s) = \delta(s-\tau)$, $w(x,t) = u(x,t-\tau)$ and then the memory-based diffusion takes the form of a discrete delay $d_{21}div(v(x,t)\nabla u(x,t-\tau))$. The corresponding case of discrete delay in system (1.1) has been considered in [22,23], where the conditions for stability and Hopf bifurcation and the normal forms of spatially inhomogeneous Hopf bifurcations are derived.

In this paper, we are interested in the frequently encountered weak delay kernel in the literature on delay equations [3,8], taking as

$$F(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \ \tau > 0. \tag{1.3}$$

This kernel function is strictly monotonically decreasing with respect to the variable t, which reflects that the memory of animals can become ambiguous over time. It follows from (1.3) that $\int_0^{+\infty} tF(t)dt = \tau$. Thus, in what follows we call τ the mean delay for the kernel function (1.3). In addition, it is easy to verify that $\lim_{\tau \to 0^+} F(t) = 0$ and $\lim_{\tau \to +\infty} F(t) = 0$, which, together with (1.2), implies that for the kernel function (1.3), $\lim_{\tau \to 0^+} w(x,t) = 0$ and $\lim_{\tau \to +\infty} w(x,t) = 0$. This means that there is no memory-based diffusion when there is no memory or memory is too old. Therefore, we are interested in the case of $\tau \in (0, +\infty)$ and mainly investigate the influence of the mean delay τ on the stability of system (1.1) and the corresponding bifurcation phenomena. Our main findings are summarized as follows:

(i) The influence of the mean delay τ on the stability of the positive constant steady state of system (1.1) and the conditions of the occurrence of Turing bifurcation and Hopf bifurcation are investigated;
(ii) For $d_{21} < 0$ (the toxic resources), d_{21} can induce the Turing bifurcation and the mean delay τ does not affect the stability of the positive constant steady state;
(iii) For $d_{21} > 0$ (the available resources), there exists a threshold d_H^* such that the mean delay τ does not affect the stability of the positive constant steady state for $d_{21} < d_H^*$ and can induce the Hopf bifurcation for $d_{21} > d_H^*$, and when $d_{21} > d_H^*$, there exist two critical

values τ_* and τ^* of τ such that the positive constant steady state is asymptotically stable for $\tau \in (0, \tau_*) \cup (\tau^*, \infty)$ and unstable for $\tau \in (\tau_*, \tau^*)$;

(iv) The properties of Turing bifurcation are considered and the normal form associated with the Hopf bifurcation is derived.

The rest of the paper is organized as follows: In Section 2, we investigate the stability of the positive constant steady state, and derive the conditions for the Turing bifurcation and delay-induced Hopf bifurcation. In Section 3, we first derive the equivalent system of (1.1) with the weak kernel, and then the direction and stability of bifurcation are illustrated. We apply the obtained theoretical results to a consumer-resource model with distributed delay and Holling type-II functional response, and the properties of steady-state solution and periodic solution are determined in Section 4. We conclude and discuss our work in Section 5, and some detailed proofs are given in the Appendixes A & B. We denote by $\mathbb{N}$ the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Linear stability and bifurcation analysis

In this section, we consider the linear stability and possible bifurcation induced by the memory-based diffusion coefficient d_{21} and the mean delay τ for the positive constant steady state of system (1.1).

Let (u_*, v_*) be a positive constant steady state of system (1.1). Then the linearized system of (1.1) at (u_*, v_*) is

$$\begin{pmatrix} u_t(x,t) \\ v_t(x,t) \end{pmatrix} = D_1 \begin{pmatrix} \Delta u(x,t) \\ \Delta v(x,t) \end{pmatrix} + D_2 \begin{pmatrix} \Delta w(x,t) \\ \Delta v(x,t) \end{pmatrix} + A \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}, \tag{2.1}$$

where

$$D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \; D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21} v_* & 0 \end{pmatrix}, \; A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.2}$$

and

$$a_{11} = f'_u(u_*, v_*), a_{12} = f'_v(u_*, v_*), a_{21} = g'_u(u_*, v_*), a_{22} = g'_v(u_*, v_*).$$

For the biological meaning of the consumer-resource model, in what follows, we give the following basic assumption

$$(C_1) \qquad a_{12} < 0, \; a_{21} > 0.$$

Considering the Neumann boundary condition, let $0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_n \le \cdots \to +\infty$ as $j \to +\infty$, be the eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta \gamma(x) + \sigma \gamma(x) = 0, & x \in \Omega, \\ \dfrac{\partial \gamma(x)}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

and $\gamma_n(x)$ be the normalized corresponding eigenfunctions of σ_n. Then assume that the solution of Eq. (2.1) is in form of

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} A_n \\ B_n \end{pmatrix} e^{\lambda_n t}\gamma_n(x). \tag{2.4}$$

We have the following characteristic equation of linearized system (2.1)

$$\lambda^2 - T_n\lambda + J_n - d_{21}v_* a_{12}\sigma_n \int_0^{+\infty} F(s)e^{-\lambda s}ds = 0,\ n \in \mathbb{N}_0, \tag{2.5}$$

where

$$\begin{aligned} T_n &= Tr(A) - Tr(D_1)\sigma_n, \\ J_n &= d_{11}d_{22}\sigma_n^2 - (d_{11}a_{22} + d_{22}a_{11})\sigma_n + Det(A), \end{aligned} \tag{2.6}$$

with

$$Tr(A) = a_{11} + a_{22},\ Tr(D_1) = d_{11} + d_{22},\ Det(A) = a_{11}a_{22} - a_{12}a_{21}. \tag{2.7}$$

When $d_{21} = 0$, (2.5) becomes $\lambda^2 - T_n\lambda + J_n = 0,\ n \in \mathbb{N}_0$. In order to investigate the influence of the memory-based diffusion on the stability of (u_*, v_*) of (1.1), we assume that there is no random-diffusion-driven Turing instability for system (1.1) without memory-driven diffusion ($d_{21} = 0$). For this purpose, we assume that

$$(C_2) \qquad Tr(A) < 0,\ Det(A) > 0,$$

and

$$(C_3) \qquad d_{11}a_{22} + d_{22}a_{11} < 2\sqrt{d_{11}d_{22}Det(A)}$$

hold. And it is easy to see from (2.6) and (2.7) that under these two assumptions (C_2) and (C_3), $T_n < 0$ and $J_n > 0$ for any $n \in \mathbb{N}_0$. This implies that the positive constant steady state (u_*, v_*) of (1.1) without memory-driven diffusion ($d_{21} = 0$) is locally asymptotically stable for any $d_{11}, d_{22} \geq 0$.

When $d_{21} \neq 0$ and $\tau = 0$, (2.5) becomes

$$\lambda^2 - T_n\lambda + J_n - d_{21}v_* a_{12}\sigma_n = 0,\ \ n \in \mathbb{N}_0, \tag{2.8}$$

because $\lim_{\tau\to 0^+} \int_0^{+\infty} F(t)e^{-\lambda s}ds \overset{\frac{s}{\tau}=\eta}{=} \lim_{\tau\to 0^+} \int_0^{+\infty} e^{-\eta}e^{-\lambda\tau\eta}d\eta = \int_0^{+\infty} e^{-\eta}d\eta = 1$. In terms of the assumptions (C_1), (C_2) and (C_3), we have the fact that there exist critical values $d_{21,n}^S$ defined by

$$d_{21,n}^S = \frac{J_n}{v_* a_{12}\sigma_n} < 0, \qquad n \in \mathbb{N}, \tag{2.9}$$

such that $J_n - d_{21}v_*a_{12}\sigma_n > 0$ for $d_{21} > d^S_{21,n}$ and $J_n - d_{21}v_*a_{12}\sigma_n \le 0$ for $d_{21} \le d^S_{21,n}$. Therefore, without memory ($\tau = 0$), the positive diffusion coefficient $d_{21}(> 0)$ does not affect the stability of (u_*, v_*) and the negative diffusion coefficient $d_{21}(< 0)$ leads to the occurrence of Turing bifurcations to be discussed in detail later.

In what follows, we restrict our attention to the case of $d_{21} \ne 0$ and $\tau > 0$. For the weak kernel (1.3), we have

$$\int_0^{+\infty} F(s)e^{-\lambda s}ds = \int_0^{+\infty} \frac{1}{\tau}e^{-\left(\lambda+\frac{1}{\tau}\right)s}ds = \begin{cases} \frac{1}{1+\tau\lambda}, & \mathrm{Re}\lambda + \frac{1}{\tau} > 0, \\ +\infty, & \mathrm{Re}\lambda + \frac{1}{\tau} \le 0, \end{cases}$$

which, together with (2.5), implies that for $n \in \mathbb{N}$, Eq. (2.5) has no roots satisfying $\mathrm{Re}\lambda \le -\frac{1}{\tau}$. When $\mathrm{Re}\lambda > -\frac{1}{\tau}$, Eq. (2.5) is equivalent to the following equation

$$\lambda^3 + P_n\lambda^2 + Q_n\lambda + R_n = 0, \tag{2.10}$$

where

$$P_n = \frac{1}{\tau} - T_n, \; Q_n = J_n - \frac{T_n}{\tau}, \; R_n = \frac{J_n - d_{21}v_*a_{12}\sigma_n}{\tau}. \tag{2.11}$$

Taking d_{21} and τ as parameters, we investigate the distribution of roots of Eq. (2.10), which determines the stability of (u_*, v_*). Obviously, P_n and Q_n are both positive with the conditions (C_1), (C_2) and (C_3). Applying the Routh-Hurwitz criterion, the following results are established immediately.

Proposition 2.1. *Under the conditions (C_1), (C_2) and (C_3), we have:*

(i) All roots of Eq. (2.10) have negative real part if and only if $R_n > 0$ and $P_nQ_n - R_n > 0$;
(ii) If $R_n = 0$, then Eq. (2.10) has a simple zero root and two roots with negative real part;
(iii) Eq. (2.10) has a pair of purely imaginary roots $\pm i\sqrt{Q_n}$ and a negative real root if and only if $P_nQ_n - R_n = 0$.

In terms of Proposition 2.1, the stability and possible bifurcation of (u_*, v_*) are related to the signs of R_n and $P_nQ_n - R_n$. From (2.9) and (2.11), the following results on the sign of R_n are obvious.

Proposition 2.2. *Assume that the conditions (C_1), (C_2), and (C_3) hold and $d^S_{21,n}$ is defined by (2.9). Then, for R_n, we have the following results:*

(i) If $d_{21} \ge 0$, then $R_n > 0$ for any $n \in \mathbb{N}$;
(ii) If $d_{21} < 0$, then

$$R_n \begin{cases} > 0, & d_{21} > d^S_{21,n}, \\ = 0, & d_{21} = d^S_{21,n}, \\ < 0, & d_{21} < d^S_{21,n}. \end{cases} \tag{2.12}$$

From (2.11), we have

$$P_n Q_n - R_n = \frac{-T_n J_n \tau^2 + (T_n^2 + d_{21} v_* a_{12}\sigma_n)\tau - T_n}{\tau^2}. \tag{2.13}$$

In terms of Propositions 2.1 and 2.2 and taking d_{21} as a bifurcation parameter, we have following results on the distribution of zero roots of Eq. (2.10), which is independent of τ.

Lemma 2.3. *Assume that the conditions* (C_1), (C_2) *and* (C_3) *hold and* $d_{21,n}^S$ *is defined by* (2.9), *and define*

$$d_S^* = \max_{n\in\mathbb{N}}\{d_{21,n}^S\}. \tag{2.14}$$

Then the following statements hold.

(i) $\lambda = 0$ *is a root of Eq.* (2.10) *if and only if* $d_{21} = d_{21,n}^S$;
(ii) *When* $d_S^* < d_{21} \le 0$, *all roots of Eq.* (2.10) *have negative real parts; and when* $d_{21} < d_S^*$, *Eq.* (2.10) *has at least one positive root.*

Proof. By Proposition 2.1 and (2.12), the conclusion (i) follows immediately.
Since

$$d_{21,n}^S = \frac{J_n}{v_* a_{12}\sigma_n} = \frac{1}{v_* a_{12}}\left(d_{11}d_{22}\sigma_n + \frac{Det(A)}{\sigma_n} - (d_{11}a_{22} + d_{22}a_{11})\right),$$

and note the fact that σ_n increases in n and $\sigma_n \to \infty$ as $n \to \infty$. Hence, it is easy to verify that $d_{21,n}^S$ is increasing for $\sigma_n < \sqrt{\frac{Det(A)}{d_{11}d_{22}}}$, decreasing for $\sigma_n > \sqrt{\frac{Det(A)}{d_{11}d_{22}}}$ and $d_{21,n}^S \to -\infty$ as $n \to +\infty$, which implies that $d_S^* = \max_{n\in\mathbb{N}}\{d_{21,n}^S\}$ exists.

It is easy to conclude that for $d_{21} \le 0$, $-T_n J_n \tau^2 + (T_n^2 + d_{21} v_* a_{12}\sigma_n)\tau - T_n > 0$ because $T_n < 0$, $J_n > 0$ and $a_{12} < 0$. This, together with (2.13), implies that $P_n Q_n - R_n > 0$ for $d_{21} \le 0$. Therefore, when $d_S^* < d_{21} \le 0$, we have $R_n > 0$ and $P_n Q_n - R_n > 0$ for any $n \in \mathbb{N}$. This, together with Proposition 2.1, implies that all roots of Eq. (2.10) have negative real parts for $d_S^* < d_{21} \le 0$. For fixed $d_{21} < d_S^*$, by the definition of d_S^* and (2.12), there exists at least one positive integer n such that $R_n < 0$, which implies that Eq. (2.10) has at least one positive root. This completes the proof of conclusion (ii). □

In addition, we have the fact that the transversality condition holds at the critical valves $d_{21} = d_{21,n}^S$ as follows.

Lemma 2.4. *Letting* $\lambda(d_{21})$ *be the root of Eq.* (2.10) *around* $d_{21} = d_{21,n}^S$ *satisfying* $\lambda(d_{21,n}^S) = 0$, *where* $d_{21,n}^S$ *is defined by* (2.9). *Then*

$$\left.\frac{\mathrm{d}\lambda(d_{21})}{\mathrm{d}d_{21}}\right|_{d_{21}=d_{21,n}^S} < 0.$$

Proof. Taking λ as the function of d_{21}, and differentiating of Eq. (2.10) with respect to d_{21}, we have

$$\frac{\mathrm{d}\lambda(d_{21})}{\mathrm{d}d_{21}} = \frac{v_* a_{12}\sigma_n}{\tau\left(3\lambda^2 + 2(\frac{1}{\tau} - T_n) + J_n - \frac{T_n}{\tau}\right)},$$

which implies that

$$\left.\frac{\mathrm{d}\lambda(d_{21})}{\mathrm{d}d_{21}}\right|_{d_{21}=d^S_{21,n}} = \frac{v_* a_{12}\sigma_n}{\tau\left(2(\frac{1}{\tau} - T_n) + J_n - \frac{T_n}{\tau}\right)} < 0,$$

since $a_{12}, T_n < 0$, $J_n > 0$. This completes the proof. □

In the following, we investigate the distribution of roots of Eq. (2.10) for $d_{21} > 0$. We first have the results on the sign of $P_n Q_n - R_n$.

Proposition 2.5. *Assume that the conditions (C_1), (C_2) and (C_3) hold. Letting*

$$\hat{d}_{21}(\sigma_n) = \frac{-T_n^2}{v_* a_{12}\sigma_n} + \frac{2T_n\sqrt{J_n}}{v_* a_{12}\sigma_n}, \tag{2.15}$$

then, for $P_n Q_n - R_n$, we have the following results:

(i) when $0 < d_{21} < \hat{d}_{21}(\sigma_n)$, $P_n Q_n - R_n > 0$ for any $\tau \geq 0$;
(ii) when $d_{21} = \hat{d}_{21}(\sigma_n)$, $P_n Q_n - R_n > 0$ for $\tau \in [0, \tau_n) \cup (\tau_n, \infty)$ and $P_n Q_n - R_n = 0$ at $\tau = \tau_n$,

$$\tau_n = \frac{T_n^2 + d_{21} v_* a_{12}\sigma_n}{2T_n J_n} = \frac{1}{\sqrt{J_n}}; \tag{2.16}$$

(iii) when $d_{21} > \hat{d}_{21}(\sigma_n)$, $P_n Q_n - R_n > 0$ for $\tau \in [0, \tau_n^+) \cup (\tau_n^-, \infty)$ and $P_n Q_n - R_n = 0$ at $\tau = \tau_n^+$ or $\tau = \tau_n^-$, where

$$\tau_n^+ = \frac{T_n^2 + d_{21} v_* a_{12}\sigma_n + \sqrt{M_n}}{2T_n J_n}, \quad \tau_n^- = \frac{T_n^2 + d_{21} v_* a_{12}\sigma_n - \sqrt{M_n}}{2T_n J_n}, \quad 0 < \tau_n^+ < \tau_n^-, \tag{2.17}$$

with

$$M_n = (T_n^2 + d_{21} v_* a_{12}\sigma_n)^2 - 4T_n^2 J_n = v_*^2 a_{12}^2 \sigma_n^2 d_{21}^2 + 2v_* a_{12}\sigma_n T_n^2 d_{21} + T_n^4 - 4T_n^2 J_n. \tag{2.18}$$

Proof. It follows from (2.13) that $P_n Q_n - R_n \geq 0$ is equivalent to

$$-T_n J_n \tau^2 + (T_n^2 + d_{21} v_* a_{12}\sigma_n)\tau - T_n \geq 0. \tag{2.19}$$

Notice that $(T_n^2 + d_{21}v_*a_{12}\sigma_n) \geq 0$ is equivalent to $d_{21} \leq \tilde{d}_{21}(\sigma_n)$. And then from (2.19) and the fact that $T_n < 0$ and $a_{12} < 0$, it is easy to see that when $d_{21} \leq \tilde{d}_{21}(\sigma_n)$, $P_nQ_n - R_n > 0$ for any $\tau \geq 0$, where

$$\tilde{d}_{21}(\sigma_n) = \frac{T_n^2}{-v_*a_{12}\sigma_n}. \tag{2.20}$$

From (2.15) and (2.20) and noticing that $T_n < 0$ and $a_{12} < 0$, we have

$$\tilde{d}_{21}(\sigma_n) < \hat{d}_{21}(\sigma_n).$$

From (2.18), it is easy to verify that $M_n < 0$ for $\tilde{d}_{21}(\sigma_n) < d_{21} < \hat{d}_{21}(\sigma_n)$ and $M_n \geq 0$ for $d_{21} \geq \hat{d}_{21}(\sigma_n)$. Thus, when $\tilde{d}_{21}(\sigma_n) < d_{21} < \hat{d}_{21}(\sigma_n)$, $P_nQ_n - R_n > 0$ for any $\tau \geq 0$. This confirms (i).

Noticing that $T_n < 0$ and $a_{12} < 0$ and when $d_{21} \geq \hat{d}_{21}(\sigma_n)$, $M_n \geq 0$ and $(T_n^2 + d_{21}v_*a_{12}\sigma_n) < 0$, (ii) and (iii) are obviously true. □

From Propositions 2.1, 2.2 and 2.5, the following results on the distribution of roots of Eq. (2.10) for $d_{21} > 0$ follows immediately.

Lemma 2.6. *Assume that the conditions (C_1), (C_2) and (C_3) hold and $\hat{d}_{21}(\sigma_n)$, τ_n and $\tau_n^{\pm}$ are defined by (2.15), (2.16) and (2.17), respectively. Then, for fixed $n \in \mathbb{N}$, we have the following results:*

(i) when $0 < d_{21} < \hat{d}_{21}(\sigma_n)$, three roots of Eq. (2.10) have negative real parts for any $\tau \geq 0$;
(ii) when $d_{21} = \hat{d}_{21}(\sigma_n)$, three roots of Eq. (2.10) have negative real parts for $\tau \neq \tau_n$, and Eq. (2.10) has a pair of purely imaginary roots $\pm i\omega_n$ and one negative root at $\tau = \tau_n$;
(iii) when $d_{21} > \hat{d}_{21}(\sigma_n)$, Eq. (2.10) has a pair of purely imaginary roots $\pm i\omega_n^+$ (resp. $\pm i\omega_n^-$) and one negative root at $\tau = \tau_n^+$ (resp. $\tau = \tau_n^-$), where

$$\omega_n^{\pm} = \left(J_n - \frac{T_n}{\tau_n^{\pm}}\right)^{\frac{1}{2}}. \tag{2.21}$$

In the following, we investigate the monotonicity of $\hat{d}_{21}(\sigma_n)$ with respective to σ_n so that we can determine the global distribution of roots of the characteristic equation (2.10) for any $n \in \mathbb{N}$.

From (2.15) and (2.20), we have

$$\hat{d}_{21}(\sigma_n) = \tilde{d}_{21}(\sigma_n) + H(\sigma_n), \tag{2.22}$$

where

$$H(\sigma_n) = \frac{2T_n\sqrt{J_n}}{v_*a_{12}\sigma_n}. \tag{2.23}$$

If we treat σ_n as continuous variable and notice that T_n and J_n are both functions of σ_n, then we have

$$\frac{\mathrm{d}\tilde{d}_{21}(\sigma_n)}{\mathrm{d}\sigma_n} = -\frac{1}{a_{12}v_*}\left(Tr^2(D_1) - \frac{Tr^2(A)}{\sigma_n^2}\right)\begin{cases} > 0, & \sigma_n > \frac{-Tr(A)}{Tr(D_1)}, \\ \le 0, & \sigma_n \le \frac{-Tr(A)}{Tr(D_1)}. \end{cases} \tag{2.24}$$

In the following, we investigate the monotonicity of $H(\sigma_n)$ with respective to σ_n, which is a little bit complicated. From (2.6) and (2.23), we have

$$\frac{\mathrm{d}H(\sigma_n)}{\mathrm{d}\sigma_n} = \frac{-1}{v_*a_{12}\sigma_n^2\sqrt{J_n}}\left(Tr(A)\left(2Det(A) - (d_{11}a_{22} + d_{22}a_{11})\sigma_n\right) + Tr(D_1)\sigma_n^2 J_n'\right), \tag{2.25}$$

where J_n' is the derivative of J_n with respective to σ_n, i.e.,

$$J_n' = 2d_{11}d_{22}\sigma_n - (d_{11}a_{22} + d_{22}a_{11}). \tag{2.26}$$

Then we can prove the following result.

Proposition 2.7. *Assume that the conditions (C_1), (C_2), and (C_3) hold. There exists a positive number $\sigma_* > 0$ such that $\frac{\mathrm{d}H(\sigma_n)}{\mathrm{d}\sigma_n} > 0$ for $\sigma_n > \sigma_*$, where*

$$\sigma_* = \begin{cases} \max\left\{\frac{2Det(A)}{d_{11}a_{22}+d_{22}a_{11}}, \frac{d_{11}a_{22}+d_{22}a_{11}}{2d_{11}d_{22}}\right\}, & d_{11}a_{22} + d_{22}a_{11} > 0, \\ \sqrt[3]{-\frac{Tr(A)Det(A)}{d_{11}d_{22}Tr(D_1)}}, & d_{11}a_{22} + d_{22}a_{11} = 0, \\ \sigma_* \text{ is the unique positive root of } h(\sigma) = 0, & d_{11}a_{22} + d_{22}a_{11} < 0, \end{cases} \tag{2.27}$$

with

$$\begin{aligned} h(\sigma) \;\; = \;\; & 2Tr(D_1)d_{11}d_{22}\sigma^3 - (d_{11}a_{22} + d_{22}a_{11})Tr(D_1)\sigma^2 \\ & -(d_{11}a_{22} + d_{22}a_{11})Tr(A)\sigma + 2Tr(A)Det(A). \end{aligned} \tag{2.28}$$

Proof. We prove Proposition 2.7 according to the following three cases.

(*i*) For $d_{11}a_{22} + d_{22}a_{11} > 0$, it follows from (2.26) that $J_n' > 0$ provided that $\sigma_n > \frac{d_{11}a_{22}+d_{22}a_{11}}{2d_{11}d_{22}}$. In addition, notice that $a_{12} < 0$, $Tr(A) < 0$ and $2Det(A) - (d_{11}a_{22} + d_{22}a_{11})\sigma_n < 0$ for $\sigma_n > \frac{2Det(A)}{d_{11}a_{22}+d_{22}a_{11}}$. Therefore, letting

$$\sigma_* = \max\left\{\frac{2Det(A)}{d_{11}a_{22} + d_{22}a_{11}}, \frac{d_{11}a_{22} + d_{22}a_{11}}{2d_{11}d_{22}}\right\},$$

then by (2.25) we complete the proof for $d_{11}a_{22} + d_{22}a_{11} > 0$.

(*ii*) For $d_{11}a_{22} + d_{22}a_{11} = 0$, it is easy from (2.25) and (2.26) to verify that $\frac{\mathrm{d}H(\sigma_n)}{\mathrm{d}\sigma_n} > 0$ for $\sigma_n > \sigma_*$ if we set

$$\sigma_* = \sqrt[3]{-\frac{Tr(A)Det(A)}{d_{11}d_{22}Tr(D_1)}}.$$

(*iii*) For $d_{11}a_{22} + d_{22}a_{11} < 0$, it follows from (2.26) that $J_n' > 0$ for any $\sigma_n \geq 0$. Substituting (2.26) into (2.25), we also have

$$\begin{aligned}\frac{\mathrm{d}H(\sigma_n)}{\mathrm{d}\sigma_n} &= \frac{-1}{v_* a_{12}\sigma_n^2\sqrt{J_n}}\big(2Tr(D_1)d_{11}d_{22}\sigma_n^3 - (d_{11}a_{22}+d_{22}a_{11})Tr(D_1)\sigma_n^2 \\ &\quad -(d_{11}a_{22}+d_{22}a_{11})Tr(A)\sigma_n + 2Tr(A)Det(A)\big).\end{aligned} \tag{2.29}$$

It is easy to verify that the following cubic equation $h(\sigma) = 0$ for σ has a unique positive root denoted by σ_* and $h(\sigma) > 0$ for $\sigma > \sigma_*$. Then it follows from (2.29) that $\frac{\mathrm{d}H(\sigma_n)}{\mathrm{d}\sigma_n} > 0$ for $\sigma_n > \sigma_*$. □

From (2.22), (2.24) and Proposition 2.7, we can prove the following results on the monotonicity of $\hat{d}_{21}(\sigma_n)$ with respective to σ_n.

Proposition 2.8. *Assuming that the conditions (C_1), (C_2) and (C_3) hold and $\hat{d}_{21}(\sigma_n)$, σ_* are defined by* (2.15) *and* (2.27)*, respectively, then we have the following results:*

(*i*) *when $\sigma_n > \max\left\{\sigma_*, \frac{-Tr(A)}{Tr(D_1)}\right\}$, $\hat{d}_{21}(\sigma_n)$ is increasing with respective to σ_n;*
(*ii*) *letting*

$$\tilde{n} = \min\left\{n \in \mathbb{N}\,\middle|\,\sigma_n > \max\left\{\sigma_*, \frac{-Tr(A)}{Tr(D_1)}\right\}\right\}, \tag{2.30}$$

then

$$d_H^* = \min_{n\in\mathbb{N}}\left\{\hat{d}_{21}(\sigma_n)\right\} = \min_{1\leq n\leq\tilde{n}}\left\{\hat{d}_{21}(\sigma_n)\right\} \tag{2.31}$$

exists.

For fixed $d_{21} > d_H^*$, define

$$U(d_{21}) = \left\{n \in \mathbb{N}\,\middle|\,\hat{d}_{21}(\sigma_n) < d_{21}\right\}.$$

It follows from Proposition 2.8 that $U(d_{21})$ is a finite set. Define

$$\tau_* = \min_{n\in U(d_{21})} \tau_n^+, \quad \tau^* = \max_{n\in U(d_{21})} \tau_n^-. \tag{2.32}$$

Then from Propositions 2.6 and 2.8, we have the following results.

Lemma 2.9. *Assume that the conditions (C_1), (C_2) and (C_3) hold. d_H^*, τ_* and τ^* are defined by* (2.31) *and* (2.32)*, respectively. Then, we have the following results on the distribution of roots of Eq.* (2.10)*.*

(*i*) *when $0 < d_{21} < d_H^*$, all roots of Eq.* (2.10) *have negative real parts for any $\tau \geq 0$ and $n \in \mathbb{N}$;*

(ii) when $d_{21} > d_H^*$, there exist two critical values τ_* and τ^* of τ such that all roots of Eq. (2.10) have negative real parts for $\tau \in [0, \tau_*) \cup (\tau^*, +\infty)$ and any $n \in \mathbb{N}$, and there exist at least one pair of complex roots with positive real parts for $\tau \in (\tau_*, \tau^*)$ and some $n \in U(d_{21})$, and Eq. (2.10) has a pair of purely imaginary roots for some $n \in U(d_{21})$ at $\tau = \tau_*$ or $\tau = \tau^*$ and all other roots of (2.10) have negative real parts.

Then, we state that the transversality condition holds at critical delay values $\tau = \tau_n, \tau_n^{\pm}$ as follows.

Lemma 2.10. *Let $\lambda(\tau) = \alpha(\tau) \pm i\beta(\tau)$ be the pair of roots of Eq. (2.10) near $\tau = \tau_n$, $\tau = \tau_n^+$ or $\tau = \tau_n^-$ satisfying $\alpha(\tau_n, \tau_n^{\pm}) = 0$, $\beta(\tau_n) = \omega_n$ and $\beta(\tau_n^{\pm}) = \omega_n^{\pm}$. Then*

$$\left.\frac{\mathrm{dRe}(\lambda(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_n} = 0, \quad \left.\frac{\mathrm{dRe}(\lambda(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_n^+} > 0, \quad \left.\frac{\mathrm{dRe}(\lambda(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_n^-} < 0.$$

Proof. Differentiating both sides of Eq. (2.10) with respect to τ and noticing that λ is a function of τ, we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \frac{\frac{1}{\tau^2}\lambda^2 - \frac{T_n}{\tau}\lambda + \frac{J_n - d_{21}v_* a_{12}\sigma_n}{\tau^2}}{3\lambda^2 + 2(\frac{1}{\tau} - T_n)\lambda + J_n - \frac{T_n}{\tau}}.$$

Using (2.21) and (2.17), we have

$$\left.\left(\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right)\right|_{\tau=\tau_n,\tau_n^{\pm}} = \frac{T_n\left(\tau^2 J_n - 1\right)}{2\tau^3\left(\omega^2 + (\frac{1}{\tau} - T_n)^2\right)} \begin{cases} = 0, & \tau = \tau_n, \\ > 0, & \tau = \tau_n^+, \\ < 0, & \tau = \tau_n^-, \end{cases} \tag{2.33}$$

where we have used the fact that

$$\tau_n = \frac{1}{\sqrt{J_n}}, \ \tau_n^+ < \frac{1}{\sqrt{J_n}}, \ \tau_n^- > \frac{1}{\sqrt{J_n}},$$

in terms of $\tau_n^+\tau_n^- = \frac{1}{J_n}$ by (2.19). So, (2.33) together with the fact that

$$\frac{\mathrm{dRe}(\lambda(\tau))}{\mathrm{d}\tau} = \mathrm{Re}\left(\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau}\right)$$

completes the proof. □

Based on Lemmas 2.3, 2.4 2.6, 2.9 and 2.10, we obtain the following results on the stability and bifurcations of $E_* = (u_*, v_*)$ for Eq. (1.1).

Theorem 2.11. *Assume the conditions (C_1), (C_2) and (C_3) hold. $d_{21,n}^S$, d_S^*, d_H^* and $\tau_n^{\pm}$ are defined by (2.9), (2.14), (2.31) and (2.17), respectively, τ_* and τ^* are defined by (2.32). Then, we have following results.*

(i) *If $d_S^* < d_{21} < d_H^*$, then the positive constant steady state E_* of system (1.1) is locally asymptotically stable;*

(ii) *If $d_{21} < d_S^*$, then the positive constant steady state E_* of system (1.1) is unstable, and the Turing bifurcations occur at $d_{21} = d_{21,n}^S$ for $n \in \mathbb{N}$;*

(iii) *If $d_{21} > d_H^*$, there exist two critical values τ_* and τ^* of τ such that the positive constant steady state E_* of system (1.1) is locally asymptotically stable for $\tau \in [0, \tau_*) \cup (\tau^*, +\infty)$ and is unstable for $\tau \in (\tau_*, \tau^*)$, and the mode$-n$ Hopf bifurcations occur at $\tau = \tau_n^{\pm}$ for $n \in U(d_{21})$ with the emerge of the spatially inhomogeneous periodic solutions.*

Remark 2.12. It is well known that the prey-taxis ($d_{21} > 0$) stabilize the positive steady state E_* of (1.1) without delay. It has been shown in [23] that the discrete delay can destabilize the positive steady state E_* of (1.1). Theorem 2.11 shows the similar destabilized effect to the case of the discrete delay. However, for the distributed delay, the main characteristic different from the discrete delay is that the destabilized effect occurs only for the mean delay being a appropriate interval, and the mean delay still stabilizes the positive steady state E_* for the smaller or larger mean delay.

3. The properties of Turing and Hopf bifurcations

From Theorem 2.11, system (1.1) undergoes Turing bifurcation at $d_{21} = d_{21,n}^S < 0$ and undergoes Hopf bifurcation at $\tau = \tau_n^{\pm}$ for $n \in U(d_{21})$ and $d_{21} > d_H^*$. In this section, we investigate the properties of Turing and Hopf bifurcations to determine the types of bifurcation and the stability of bifurcating solutions. Motivated by the ideal of [8,21], in this section we use the following equivalent system of (1.1) to achieve the target

$$\begin{cases} u_t(x,t) = d_{11}\Delta u(x,t) + f(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ v_t(x,t) = d_{22}\Delta v(x,t) - d_{21} div(v(x,t)\nabla w(x,t)) + g(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ w_t(x,t) = \dfrac{1}{\tau}(u(x,t) - w(x,t)), & x \in \Omega, t > 0, \\ \partial_{\boldsymbol{n}} u(x,t) = \partial_{\boldsymbol{n}} v(x,t) = \partial_{\boldsymbol{n}} w(x,t) = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{3.1}$$

3.1. Properties of Turing bifurcation

Employing the method developed in [4,17] and combining with Lemmas 2.3 and 2.4, we have the following results.

Theorem 3.1. *Suppose that the conditions (C_1), (C_2) and (C_3) hold, and $d_{21,n}^S$, d_S^* are defined by (2.9), (2.14), respectively.*

(i) *For a fixed n, we assume that σ_n is a simple eigenvalue of (2.3), and $d_{21,n}^S \neq d_{21,k}^S$ for any $k \in \mathbb{N}$ and $k \neq n$. Hence, $d_{21} = d_{21,n}^S$ is a bifurcation point for system (3.1). Further more, around the point $(d_{21,n}^S, u_*, v_*, u_*)$, there is a smooth curve Γ_n of positive solutions of system (3.1) bifurcating from the line of constant solutions $\{(d_{21}, u_*, v_*, u_*) | d_{21} < 0\}$ with the following form:*

$$\Gamma_n = \{(d_{21,n}(s), U_n(s,x), V_n(s,x), W_n(s,x) | -\delta < s < \delta\}, \tag{3.2}$$

where δ is a positive constant small enough and

$$\begin{aligned} U_n(s,x) &= u_* + s\gamma_n(x) + sz_{1,n}(s,x), \\ V_n(s,x) &= v_* + s\frac{(d_{11}\sigma_n - a_{11})\gamma_n(x)}{a_{12}} + sz_{2,n}(s,x), \\ W_n(s,x) &= u_* + s\gamma_n(x) + sz_{3,n}(s,x), \end{aligned} \tag{3.3}$$

with sufficiently smooth functions $d_{21,n}(s)$, $z_{1,n}(s,x)$, $z_{2,n}(s,x)$, $z_{3,n}(s,x)$ satisfying $d_{21,n}(0) = d^S_{21,n}$ and $z_{1,n}(0,x) = z_{2,n}(0,x) = z_{3,n}(0,x) = 0$.

(ii) *In particular, for one-dimensional spatial domain $\Omega = (0, \ell\pi)$, we have $d'_{21,n}(0) = 0$ and*

$$\begin{aligned} d''_{21,n}(0) = &-\frac{f''' + r_n g'''}{2r_n v_*\sigma_n} - \frac{d^s_{21,n}(2\Theta^0_2 + 4h_n\Theta^2_1 - \Theta^2_2 - 2h_n)}{2v_*} \\ &-\frac{(f''_{20} + r_n g''_{20})(2\Theta^0_1 + \Theta^2_1) + (f''_{11} + r_n g''_{11})\left(2\Theta^0_2 + \Theta^2_2 + h_n(2\Theta^0_1 + \Theta^2_1)\right) + (f''_{02} + r_n g''_{02})h_n(2\Theta^0_2 + \Theta^2_2)}{2r_n v_*\sigma_n}, \end{aligned} \tag{3.4}$$

where Θ^0_1, Θ^0_2, Θ^2_1, Θ^2_2 are given by

$$\begin{aligned} &\Theta^0_1 = \frac{a_{22}f'' - a_{12}g''}{Det(A)}, \Theta^0_2 = \frac{a_{11}g'' - a_{21}f''}{Det(A)}, \Theta^2_1 = \frac{(a_{22} - 4d_{22}\sigma_n)f'' - a_{12}(g'' + 2d^S_{21,n}\sigma_n h_n)}{12d_{11}d_{22}\sigma_n^2 - 3Det(A)}, \\ &\Theta^2_2 = \frac{(a_{11} - 4d_{11}\sigma_n)g'' - a_{21}f'' - 2d^S_{21,n}\sigma_n\left(2v_*f'' - (a_{11} - 4d_{11}\sigma_n)h_n\right)}{12d_{11}d_{22}\sigma_n^2 - 3Det(A)}, \end{aligned} \tag{3.5}$$

and

$$f''' = \frac{f'''_{30} + 3h_n f'''_{21} + 3h_n^2 f'''_{12} + h_n^3 f'''_{03}}{2}, \quad g''' = \frac{g'''_{30} + 3h_n g'''_{21} + 3h_n^2 g'''_{12} + h_n^3 g'''_{03}}{2}, \tag{3.6}$$

$$f'' = \frac{f''_{20} + 2h_n f''_{11} + h_n^2 f''_{02}}{2}, \quad g'' = \frac{g''_{20} + 2h_n g''_{11} + h_n^2 g''_{02}}{2}, \tag{3.7}$$

with $\sigma_n = \left(\frac{n}{\ell}\right)^2$, $h_n = \frac{d_{11}\sigma_n - a_{11}}{a_{12}}$, $r_n = \frac{a_{12}}{d_{22}\sigma_n - a_{22}}$, and f''_{ij}, f'''_{ij}, g''_{ij}, g'''_{ij} $(i, j \in \mathbb{N}_0)$ are given by

$$f''_{ij} = \frac{\partial^2 f(u_*, v_*)}{\partial u^i \partial v^j}, \quad f'''_{ij} = \frac{\partial^3 f(u_*, v_*)}{\partial u^i \partial v^j}, \quad g''_{ij} = \frac{\partial^2 g(u_*, v_*)}{\partial u^i \partial v^j}, \quad g'''_{ij} = \frac{\partial^3 g(u_*, v_*)}{\partial u^i \partial v^j}.$$

Let $d^S_{21,N} = d^*_S$. When $d''_{21,N}(0) < 0$, there exists a supercritical pitchfork bifurcation at $d_{21} = d^S_{21,N} = d^*_S$ and the bifurcating steady-states are asymptotically stable; when $d''_{21,N}(0) > 0$, there exists a subcritical pitchfork bifurcation at $d_{21} = d^S_{21,N} = d^*_S$ and the bifurcating steady-states are unstable.

The proof of Theorem 3.1 is given in Appendix A. From Theorem 3.1, the pitchfork bifurcation occurs at $d_{21} = d^*_S$ and the formulas determining the stability of the bifurcating non-constant steady states are derived according to the second-order and third-order partial derivatives of $f(u, v)$ and $g(u, v)$ at (u_*, v_*).

3.2. Direction and stability of Hopf bifurcation

In this subsection, we determine the direction and stability of Hopf bifurcations for one-dimensional spatial domain $\Omega = (0, \ell\pi)$ by applying the normal form theory. The common used methods of calculating the normal form of Hopf bifurcation for the reaction-diffusion equations are the algorithm developed by Hassard et al. [9] and the one developed by Faria [6]. See [15] for using the algorithm of Hassard et al. [9] to investigate the properties of Hopf bifurcation in the ordinary differential equations with distributed delay. Although the algorithm developed by [6] is developed for the reaction-diffusion system with delay, it is still applicable for the case without delay with minor revisions. More recently, in [22], we developed the algorithm of calculating the normal form of Turing-Hopf bifurcation for the classical reaction-diffusion system without delay, where the algorithm of calculating the normal form of Hopf bifurcation is also explicitly derived for the classical reaction-diffusion system without delay and without chemotaxis terms.

In this paper, the distributed delay term in (1.1) is handled by introducing a new variable w, and then using the chain technique, (1.1) is transformed into (3.1), where no delay terms are involved. In what follows, we employ the algorithm derived in [26] with some revises because of the existence of nonlinear diffusion terms in (3.1) to calculate the normal form of Hopf bifurcation and the notations used in this subsection are the same as in [26].

Letting τ_H ($\tau_H = \tau_{n_c}^+$ or $\tau_H = \tau_{n_c}^-$) be the mode-n_c Hopf bifurcation for some $n = n_c \in \mathbb{N}$, then it follows from Lemma 2.9 that at $\tau = \tau_H$, Eq. (2.10) has a pair of purely imaginary roots $\pm i\omega_{n_c}, \omega_{n_c} > 0$. Define the real-valued Sobolev space

$$\mathscr{X} = \left\{ U = (u, v, w)^T \in \left(W^{2,2}(0, \ell\pi) \right)^3, \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0, x = 0, \ell\pi \right\},$$

with the inner product

$$[U, V] = \int_0^{\ell\pi} U^T V dx, \ \ for\ U, V \in \mathscr{X}.$$

Then we take a small perturbation of τ_H by setting $\tau = \tau_H + \mu$, $|\mu| \ll 1$ such that $\mu = 0$ correspond to the Hopf bifurcation value for system (3.1). Clearly, the positive constant equilibrium remains unchanged.

Now, transferring E_* to the origin by setting

$$(\widetilde{u}(x,t), \widetilde{v}(x,t), \widetilde{w}(x,t))^T = (u(x,t), v(x,t), w(x,t))^T - (u_*, v_*, u_*)^T,$$

and dropping the tildes for simplification of notation, system (3.1) becomes

$$\frac{dU}{dt} = d\Delta U + L(\mu)(U) + F(U), \tag{3.8}$$

where $U = (u, v, w)^T$, $d\Delta U = d_0 \Delta U + F^d(U)$, $L(\mu)(U) = L_0(U) + \widetilde{L}(U, \mu)$, and

$$F(U)=\begin{pmatrix} f(u+u_*, v+v_*) \\ g(u+u_*, v+v_*) \\ \frac{1}{\tau_H}(u-w) \end{pmatrix} - L_0(U), \tag{3.9}$$

with

$$d_0\Delta=\begin{pmatrix} d_{11}\frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & d_{22}\frac{\partial^2}{\partial x^2} & -d_{21}v_*\frac{\partial^2}{\partial x^2} \\ 0 & 0 & 0 \end{pmatrix},\ F^d(U)=-\begin{pmatrix} 0 \\ d_{21}(v_x w_x+v w_{xx}) \\ 0 \end{pmatrix}, \tag{3.10}$$

and

$$L_0=\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ \frac{1}{\tau_H} & 0 & -\frac{1}{\tau_H} \end{pmatrix},\ \widetilde{L}(U,\mu)=\begin{pmatrix} 0 \\ 0 \\ \left(\tau_H(\mu)-\frac{1}{\tau_H}\right)(u-w) \end{pmatrix}. \tag{3.11}$$

We denote $\tau_H(\mu)=\frac{1}{\tau_H+\mu}$ in (3.11), and it can be written in Taylor expansion as follows:

$$\tau_H(\mu)=\frac{1}{\tau_H+\mu}=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{\tau_H^j}\mu^{j-1}.$$

In what follows, we assume that $F(U)$ is $\mathcal{C}^k$, $k\geq 3$, smooth enough with respect to U. Noticing that μ is the perturbation parameter and treated as a new variable in the calculation of normal form, we rewrite Eq. (3.8) as the following system

$$\frac{dU}{dt}=d_0\Delta U+L_0(U)+\widetilde{F}(U,\mu), \tag{3.12}$$

where

$$\widetilde{F}(U,\mu)=F(U)+\widetilde{L}(U,\mu)+F^d(U). \tag{3.13}$$

Denoting by $\mathscr{L}(U)=d_0\Delta U+L_0(U)$, the linear system of Eq. (3.12) can be written as

$$\frac{dU}{dt}=\mathscr{L}(U). \tag{3.14}$$

It is well known that the eigenvalue problem (2.3) has eigenvalues $\sigma_n=(n/\ell)^2$, $n\in\mathbb{N}_0$ with the corresponding normalized eigenfunctions

$$\gamma_n(x)=\frac{\cos\left(\frac{nx}{\ell}\right)}{||\cos\left(\frac{nx}{\ell}\right)||_{2,2}}=\begin{cases} \frac{1}{\sqrt{\ell\pi}}, & \text{when} \quad n=0, \\ \sqrt{\frac{2}{\ell\pi}}\cos\left(\frac{nx}{\ell}\right), & \text{when} \quad n\neq 0. \end{cases} \tag{3.15}$$

Let $\beta_n^{(j)} = \gamma_n(x)e_j,\ j = 1, 2, 3$, where e_j are the unit coordinate vectors of $\mathbb{R}^3$. Then the normalized eigenfunctions $\left\{\beta_n^{(j)}\right\}_{n=1}^{\infty}$ form an orthonormal basis for $\mathscr{X}$.

Set $\mathscr{B}_n$ =span$\left\{\left[\varphi(\cdot), \beta_n^{(j)}\right]\beta_n^{(j)} \mid \varphi \in \mathscr{X},\ j = 1, 2, 3\right\}$. Then it is easy to verify that

$$L_0(\mathscr{B}_n) \subset span\left\{\beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)}\right\}, n \in \mathbb{N}_0.$$

Assume that $y(t) \in \mathbb{R}^3$ and

$$y^T(t)\begin{pmatrix}\beta_n^{(1)}\\ \beta_n^{(2)}\\ \beta_n^{(3)}\end{pmatrix} \in \mathscr{B}_n.$$

Then, on $\mathscr{B}_n$, the linearized equation (3.14) is equivalent to the ODEs on $\mathbb{R}^3$

$$\dot{y}(t) = -(n/\ell)^2 d_0 y(t) + L_0(y(t)). \tag{3.16}$$

It is obvious that the characteristic equation of linear system (3.16) is the same as the linear partial differential Eq. (3.14).

Let

$$\mathcal{M}_n = \begin{pmatrix} -d_{11}\left(\frac{n}{\ell}\right)^2 + a_{11} & a_{12} & 0\\ a_{21} & -d_{22}\left(\frac{n}{\ell}\right)^2 + a_{22} & d_{21}v_*\left(\frac{n}{\ell}\right)^2\\ \frac{1}{\tau_H} & 0 & -\frac{1}{\tau_H}\end{pmatrix} \tag{3.17}$$

be the characteristic matrix of Eq. (3.16). Further, let $\Lambda = \{i\omega_{n_c}, -i\omega_{n_c}\}$, and denote the generalized eigenspace of Eq. (3.16) associated with Λ by P and the corresponding adjoint space by P^*. Then, according to the standard adjoint theory for ODEs, $\mathbb{C}^3$ can be used to decomposed by Λ as $\mathbb{C}^3 = P \oplus Q$, where $Q = \{\varphi \in \mathbb{C}^3 : \langle\psi, \varphi\rangle = 0, \forall\psi \in P^*\}$ and $\langle\cdot,\cdot\rangle$ is defined by $\langle\psi, \varphi\rangle = \psi^T\varphi$, for $\varphi, \psi \in \mathbb{C}^3$.

Choose the dual bases Φ and Ψ of P and P^*, respectively, as follows

$$\Phi = (p, \bar{p}), \quad \Psi = \mathrm{col}\left(q^T, \overline{q}^T\right),$$

such that $\langle\Psi, \Phi\rangle_{n_c} = I_2$, where

$$p = \begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix} = \begin{pmatrix} 1\\ \frac{i\omega_{n_c}+(n_c/\ell)^2 d_{11}-a_{11}}{a_{12}}\\ \frac{1}{1+i\omega_{n_c}\tau_H}\end{pmatrix}, q = \begin{pmatrix}q_1\\ q_2\\ q_3\end{pmatrix} = \eta\begin{pmatrix} 1\\ \frac{a_{12}}{i\omega_{n_c}+(n_c/\ell)^2 d_{22}-a_{22}}\\ \frac{\tau_H d_{21}v_* a_{12}(n_c/\ell)^2}{(1+i\omega_{n_c}\tau_H)(i\omega_{n_c}+(n_c/\ell)^2 d_{22}-a_{22})}\end{pmatrix},$$

with

$$\eta = \frac{\left(i\omega_{n_c} + (n_c/\ell)^2 d_{22} - a_{22}\right)(1 + i\omega_{n_c}\tau_H)^2}{\left(2i\omega_{n_c} + (n_c/\ell)^2(d_{11} + d_{22}) - (a_{11} + a_{22})\right)(1 + i\omega_{n_c}\tau_H)^2 + \tau_H d_{21}v_* a_{12}(n_c/\ell)^2},$$

since $\mathcal{M}_{n_c} p = i\omega_{n_c} p$, $\mathcal{M}_{n_c}^T q = i\omega_{n_c} q$ and $\langle q, p\rangle = 1$.

Using the decomposition $\mathbb{C}^3 = P \oplus Q$, the phase space $\mathscr{X}$ can be decomposed as

$$\mathscr{X} = \mathscr{X}^c \oplus \mathscr{X}^s,\ \mathscr{X}^c = \mathrm{Im}\pi,\ \mathscr{X}^s = \mathrm{Ker}\pi, \tag{3.18}$$

where $dim\ \mathscr{X}^c = 2$, and $\pi : \mathscr{X} \to \mathscr{X}^c$ is the projection operator defined by

$$\pi(\phi) = \Phi \left\langle \Psi, \begin{pmatrix} \left[\phi(\cdot), \beta_{n_c}^{(1)}\right] \\ \left[\phi(\cdot), \beta_{n_c}^{(2)}\right] \\ \left[\phi(\cdot), \beta_{n_c}^{(3)}\right] \end{pmatrix} \right\rangle \gamma_{n_c}(x),\ \phi \in \mathscr{X}. \tag{3.19}$$

According to (3.18), $U = (u, v, w)^T \in \mathscr{X}$ can be decomposed as

$$U = \Phi Z \gamma_{n_c}(x) + W,\ W = (w^{(1)}, w^{(2)}, w^{(3)})^T \in \mathscr{X}^s. \tag{3.20}$$

Let $Z = (z_1(t), z_2(t)) \in \mathbb{R}^2$, then system (3.12) is equivalent to the following system

$$\begin{cases} \dot{Z} = BZ + \Psi \begin{pmatrix} \left[\widetilde{F}(\Phi Z \gamma_{n_c}(x) + W, \mu), \beta_{n_c}^{(1)}\right] \\ \left[\widetilde{F}(\Phi Z \gamma_{n_c}(x) + W, \mu), \beta_{n_c}^{(2)}\right] \\ \left[\widetilde{F}(\Phi Z \gamma_{n_c}(x) + W, \mu), \beta_{n_c}^{(3)}\right] \end{pmatrix}, \\ \dot{W} = \mathscr{L}(W) + (I - \pi)\widetilde{F}(\Phi Z \gamma_{n_c}(x) + W, \mu), \end{cases} \tag{3.21}$$

where $B = \mathrm{diag}\{i\omega_{n_c}, -i\omega_{n_c}\}$ and I is the identity operator.

Consider the formal Taylor expansion

$$\begin{aligned} &\widetilde{F}(\varphi, \mu) = \sum_{j\geq 2} \frac{1}{j!}\widetilde{F}_j(\varphi, \mu), \quad F(\varphi) = \sum_{j\geq 2} \frac{1}{j!} F_j(\varphi), \quad \widetilde{L}(\varphi, \mu) = \sum_{j\geq 1} \frac{1}{j!}\widetilde{L}_j(\varphi)\mu^j, \\ &F^d(\varphi) = \sum_{j\geq 2} \frac{1}{j!} F_j^d(\varphi), \end{aligned} \tag{3.22}$$

where $\widetilde{F}_j$, F_j, $\widetilde{L}_j$, F_j^d are the j-th Fréchet derivative of $\widetilde{F}$, F, $\widetilde{L}$, F^d, respectively.

From (3.13), we have

$$\widetilde{F}_j(U, \mu) = F_j(U) + j\mu^{j-1}\widetilde{L}_{j-1}(U) + F_j^d(U), \quad j = 2, 3\cdots. \tag{3.23}$$

Then (3.21) is written as

$$\begin{cases} \dot{Z} = BZ + \sum\limits_{j\geq 2} \frac{1}{j!} f_j^1(Z, W, \mu), \\ \dot{W} = \mathscr{L}(W) + \sum\limits_{j\geq 2} \frac{1}{j!} f_j^2(Z, W, \mu), \end{cases} \tag{3.24}$$

where

$$f_j^1(Z, W, \mu) = \Psi \begin{pmatrix} \left[\widetilde{F}_j(\Phi Z \gamma_{n_c}(x) + W, \mu), \beta_{n_c}^{(1)}\right] \\ \left[\widetilde{F}_j(\Phi Z \gamma_{n_c}(x) + W, \mu), \beta_{n_c}^{(2)}\right] \\ \left[\widetilde{F}_j(\Phi Z \gamma_{n_c}(x) + W, \mu), \beta_{n_c}^{(3)}\right] \end{pmatrix}, \tag{3.25}$$

$$f_j^2(Z, W, \mu) = (I - \pi)\widetilde{F}_j(\Phi Z \gamma_{n_c}(x) + W, \mu). \tag{3.26}$$

In terms of the normal form theory of autonomous ODEs in the finite dimension space [6], after a recursive transformation of variables of the form

$$(Z, W) = (\widetilde{Z}, \widetilde{W}) + \frac{1}{j!}(U_j^1(\widetilde{Z}, \mu), U_j^2(\widetilde{Z}, \mu)), \ j \geq 2, \tag{3.27}$$

where $Z, \widetilde{Z} \in \mathbb{R}^2$, $W, \widetilde{W} \in \mathscr{X}^s$ and $U_j^1 : \mathbb{R}^3 \to \mathbb{R}^2$, $U_j^2 : \mathbb{R}^3 \to \mathscr{X}^s$ are homogeneous polynomials of degree j in $\widetilde{Z}$ and μ, then the flow on the local center manifold for Eq. (3.12) can be written as

$$\dot{Z} = BZ + \sum_{j \geq 2} \frac{1}{j!} g_j^1(Z, 0, \mu), \tag{3.28}$$

which is the normal form as in the usual sense for ODEs.

Denote the operators M_j^1 and M_j^2 by

$$\begin{aligned} &M_j^1 : V_j^3(\mathbb{C}^2) \to V_j^3(\mathbb{C}^2), M_j^1(U_j^1) = D_Z U_j^1(Z, \mu)BZ - BU_j^1(Z, \mu), \\ &M_j^2 : V_j^3(\mathscr{X}^s) \to V_j^3(\mathscr{X}^s), M_j^2(U_j^2) = D_Z U_j^2(Z, \mu)BZ - \mathscr{L}\left(U_j^2(Z, \mu)\right), \end{aligned} \tag{3.29}$$

where $V_j^3(Y)$ denotes the space of homogeneous polynomials of degree j in three variables $z_1(t), z_2(t), \mu$ with coefficients in Y. Then

$$Ker(M_2^1) = span\left\{\begin{pmatrix} z_1\mu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\mu \end{pmatrix}\right\}, \tag{3.30}$$

$$Ker(M_3^1) = span\left\{\begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1\mu^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1^2 z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\mu^2 \end{pmatrix}\right\}. \tag{3.31}$$

Similar to [22], we denote

$$f_2^{(1,1)}(Z, W, \mu) = \Psi \begin{pmatrix} \left[F_2(\Phi Z \gamma_{n_c}(x) + W) + 2\mu \widetilde{L}_1(\Phi Z \gamma_{n_c}(x) + W), \beta_{n_c}^{(1)}\right] \\ \left[F_2(\Phi Z \gamma_{n_c}(x) + W) + 2\mu \widetilde{L}_1(\Phi Z \gamma_{n_c}(x) + W), \beta_{n_c}^{(2)}\right] \\ \left[F_2(\Phi Z \gamma_{n_c}(x) + W) + 2\mu \widetilde{L}_1(\Phi Z \gamma_{n_c}(x) + W), \beta_{n_c}^{(3)}\right] \end{pmatrix}, \tag{3.32}$$

$$f_2^{(1,2)}(Z,W)=\Psi\begin{pmatrix}\left[F_2^d(\Phi Z\gamma_{n_c}(x)+W),\beta_{n_c}^{(1)}\right]\\ \left[F_2^d(\Phi Z\gamma_{n_c}(x)+W),\beta_{n_c}^{(2)}\right]\\ \left[F_2^d(\Phi Z\gamma_{n_c}(x)+W),\beta_{n_c}^{(3)}\right]\end{pmatrix}. \tag{3.33}$$

From [6] and [7], we have

$$\begin{aligned}&g_2^1(Z,0,\mu)=\mathrm{Proj}_{\mathrm{Ker}(M_2^1)}f_2^1(Z,0,\mu),\\ &g_3^1(Z,0,\mu)=\mathrm{Proj}_{\mathrm{Ker}(M_3^1)}\widetilde{f}_3^1(Z,0,\mu)=\mathrm{Proj}_{\mathrm{S}}\widetilde{f}_3^1(Z,0,0)+O(\mu^2|Z|),\end{aligned} \tag{3.34}$$

where

$$S=span\left\{\begin{pmatrix}z_1^2z_2\\0\end{pmatrix},\begin{pmatrix}0\\z_1^2z_2\end{pmatrix}\right\}, \tag{3.35}$$

and $\widetilde{f}_3^1(Z,0,\mu)$ is the term of order 3 obtained from (3.27) after the change of variables in previous step given by

$$\begin{aligned}\widetilde{f}_3^1(Z,0,\mu)=f_3^1(Z,0,\mu)+\frac{3}{2}\Big[&\left(D_Zf_2^1(Z,0,\mu)\right)U_2^1(Z,\mu)+\left(D_Wf_2^{(1,1)}(Z,0,\mu)\right)U_2^2(Z,\mu)\\ &+\left(D_{W,W_x,W_{xx}}f_2^{(1,2)}(Z,0)\right)U_2^{(2,d)}(Z,\mu)-D_ZU_2^1(Z,\mu)g_2^1(Z,0,\mu)\Big],\end{aligned} \tag{3.36}$$

where

$$f_2^1(Z,0,\mu)=f_2^{(1,1)}(Z,0,\mu)+f_2^{(1,2)}(Z,0),$$

$$D_{W,W_x,W_{xx}}f_2^{(1,2)}(Z,0,\mu)=\left(D_Wf_2^{(1,2)}(Z,0,\mu),D_{W_x}f_2^{(1,2)}(Z,0,\mu),D_{W_{xx}}f_2^{(1,2)}(Z,0,\mu)\right),$$

and

$$\begin{aligned}&U_2^1(Z,0)=\left(M_2^1\right)^{-1}\mathrm{Proj}_{\mathrm{Im}(M_2^1)}f_2^1(Z,0,0),\quad U_2^2(Z,0)=\left(M_2^2\right)^{-1}f_2^2(Z,0,0),\\ &U_2^{(2,d)}(Z,\mu)=\mathrm{col}\left(U_2^2(Z,\mu),U_{2x}^2(Z,\mu),U_{2xx}^2(Z,\mu)\right).\end{aligned} \tag{3.37}$$

For notational convenience, in what follows we let

$$\mathcal{H}(\alpha z_1^{m_1}z_2^{m_2}\mu^{m_3})=\begin{pmatrix}\alpha z_1^{m_1}z_2^{m_2}\mu^{m_3}\\ \bar{\alpha}z_1^{m_2}z_2^{m_1}\mu^{m_3}\end{pmatrix},\alpha\in\mathbb{C},m_j\in\mathbb{N}_0\text{ for }j=1,2,3.$$

We then calculate $g_j^1(Z,0,\mu)$, $j=2,3$ as follows.

3.2.1. Calculation of $g_2^1(Z,0,\mu)$

By (3.10) and (3.22), we have

$$F_2^d(U) = -2d_{21}(0, v_x w_x + v w_{xx}, 0)^T, \ F_j^d(U) = (0,0,0)^T, \ j = 3,4,\cdots, \tag{3.38}$$

and from (3.11) and (3.23), we obtain

$$\widetilde{L}_j(U) = \left(0, 0, (-1)^j \frac{1}{\tau_H^{j+1}}(u-w)\right)^T, \ j \geq 1. \tag{3.39}$$

Clearly, since for $n \in \mathbb{N}$,

$$\int_0^{\ell\pi} \gamma_n^2(x)dx = 1,$$

we can calculate that

$$\begin{pmatrix} \left[2\mu\widetilde{L}_1(\Phi Z\gamma_{n_c}(x)), \beta_{n_c}^{(1)}\right] \\ \left[2\mu\widetilde{L}_1(\Phi Z\gamma_{n_c}(x)), \beta_{n_c}^{(2)}\right] \\ \left[2\mu\widetilde{L}_1(\Phi Z\gamma_{n_c}(x)), \beta_{n_c}^{(3)}\right] \end{pmatrix} = -\frac{2\mu}{\tau_H^2}\begin{pmatrix} 0 \\ 0 \\ (p_1 - p_3)z_1 + (\overline{p_1} - \overline{p_3})z_2 \end{pmatrix}. \tag{3.40}$$

This, together with (3.13), (3.23), (3.25) and (3.39), yields to

$$g_2^1(Z,0,\mu) = \text{Proj}_{\text{Ker}(M_2^1)} f_2^1(Z,0,\mu) = \mathcal{H}(B_1 z_1 \mu), \tag{3.41}$$

where

$$B_1 = -\frac{2}{\tau_H^2} q_3(p_1 - p_3). \tag{3.42}$$

3.2.2. Calculation of $g_3^1(Z,0,\mu)$

Note that the terms $O(\mu^2|Z|)$ in (3.34) are irrelevant to determine the generic Hopf bifurcation. Thus, it is sufficient for determining the dynamics of generic Hopf bifurcation to obtain $g_3^1(Z,0,0)$ in terms of (3.34). It follows from (3.41) that $g_2^1(Z,0,0) = \mathbf{0}$. Thus, by (3.36), the term $g_3^1(Z,0,0)$ can be shown as follows:

$$\begin{aligned} g_3^1(Z,0,0) = \text{Proj}_{\text{S}} f_3^1(Z,0,0) + \frac{3}{2}\text{Proj}_{\text{S}}\Big[& \left(D_Z f_2^1(Z,0,0)\right) U_2^1(Z,0) \\ & + \left(D_W f_2^{(1,1)}(Z,0,0)\right) U_2^2(Z,0) \\ & + \left(D_{W,W_x,W_{xx}} f_2^{(1,2)}(Z,0)\right) U_2^{(2,d)}(Z,0)\Big]. \end{aligned} \tag{3.43}$$

Next, we compute $g_3^1(Z,0,0) = \mathrm{Proj}_S \widetilde{f}_3^1(Z,0,0)$ step by step according to (3.43). The calculation is divided into four steps as follows.

Step 1: The calculation of $\mathbf{Proj_S f_3^1(Z, 0, 0)}$

Let $F(U) = (F^{(1)}(U), F^{(2)}(U), F^{(3)}(U))^T$, we write

$$\frac{1}{j!}F_j(U) = \sum_{j_1+j_2+j_3=j} \frac{1}{j_1!j_2!j_3!} F_{j_1j_2j_3} u^{j_1} v^{j_2} w^{j_3}, \tag{3.44}$$

where

$$F_{j_1j_2j_3} = (F^{(1)}_{j_1j_2j_3}, F^{(2)}_{j_1j_2j_3}, 0)^T, \tag{3.45}$$

with

$$F^{(k)}_{j_1j_2j_3} = \frac{\partial^j F^k(0,0,0)}{\partial u^{j_1} \partial v^{j_2} \partial w^{j_3}},\ k = 1, 2.$$

The formal Taylor expansions of $f(u+u_*, v+v_*)$ and $g(u+u_*, v+v_*)$ at $(u, v) = (0, 0)$ can be written as

$$f(u+u_*, v+v_*) = \sum_{j_1+j_2\geq 1} \frac{1}{j_1!j_2!} f_{j_1j_2} u^{j_1} v^{j_2},\ g(u+u_*, v+v_*) = \sum_{j_1+j_2\geq 1} \frac{1}{j_1!j_2!} g_{j_1j_2} u^{j_1} v^{j_2},$$

where

$$f_{j_1j_2} = \frac{\partial^{j_1+j_2} f(u_*, v_*)}{\partial u^{j_1} \partial v^{j_2}},\ g_{j_1j_2} = \frac{\partial^{j_1+j_2} g(u_*, v_*)}{\partial u^{j_1} \partial v^{j_2}}.$$

Then, we have

$$F^{(1)}_{j_1j_2j_3} = \begin{cases} f_{j_1j_2}, & j_3 = 0, \\ 0, & j_3 \neq 0, \end{cases} \quad F^{(2)}_{j_1j_2j_3} = \begin{cases} g_{j_1j_2}, & j_3 = 0, \\ 0, & j_3 \neq 0, \end{cases}$$

which, together with (3.45), means that

$$F_{003} = F_{012} = F_{102} = F_{111} = F_{201} = F_{021} = F_{002} = F_{011} = F_{101} = \mathbf{0},$$

which will be useful in the following computation.

From (3.13), (3.38) and (3.39), we have $\widetilde{F}_3(\Phi Z \gamma_{n_c}(x), 0) = F_3(\Phi Z \gamma_{n_c}(x))$. Then, it follows from (3.25), (3.44) and the fact $\int_0^{\ell\pi} \gamma_{n_c}^4(x)dx = \frac{3}{2\ell\pi}$ that

$$\mathrm{Proj}_S f_3^1(Z,0,0) = \mathcal{H}(B_{21} z_1^2 z_2), \tag{3.46}$$

where

$$
\begin{aligned}
B_{21} &= \frac{9}{2\ell\pi} q^T \left(F_{300} p_1 |p_1|^2 + F_{030} p_2 |p_2|^2 + F_{210} \left(p_1^2 \overline{p_2} + 2 p_2 |p_1|^2 \right) + F_{120} \left(p_2^2 \overline{p_1} + 2 p_1 |p_2|^2 \right) \right) \\
&= \frac{9}{2\ell\pi} \Big((q_1 f_{30} + q_2 g_{30}) p_1 |p_1|^2 + (q_1 f_{03} + q_2 g_{03}) p_2 |p_2|^2 + (q_1 f_{21} + q_2 g_{21}) \left(p_1^2 \overline{p_2} + 2 p_2 |p_1|^2 \right) \\
&\quad + (q_1 f_{12} + q_2 g_{12}) \left(p_2^2 \overline{p_1} + 2 p_1 |p_2|^2 \right) \Big).
\end{aligned}
\tag{3.47}
$$

Step 2: The calculation of $\mathbf{Proj_S}\left(\left(\mathbf{D_z f_2^1}\right)(\mathbf{Z}, \mathbf{0}, \mathbf{0}) \mathbf{U_2^1}(\mathbf{Z}, \mathbf{0})\right)$

From (3.13), (3.38) and (3.39), we obtain

$$
\widetilde{F}_2(\Phi Z \gamma_{n_c}(x), 0) = F_2\left(\Phi Z \gamma_{n_c}(x)\right) + F_2^d(\Phi Z \gamma_{n_c}(x)). \tag{3.48}
$$

By (3.44), we write

$$
F_2(\Phi Z \gamma_{n_c}(x) + W) = \gamma_{n_c}^2(x) \left(\sum_{m_1+m_2=2} A_{m_1 m_2} z_1^{m_1} z_2^{m_2} \right) + \mathcal{S}_2(\Phi Z \gamma_{n_c}(x), W) + O(|W|^2), \tag{3.49}
$$

where

$$
\begin{aligned}
A_{20} &= F_{200} p_1^2 + 2 F_{110} p_1 p_2 + F_{020} p_2^2 = \overline{A_{02}}, \\
A_{11} &= 2 F_{200} |p_1|^2 + 4 F_{110} \mathrm{Re}\{p_1 \overline{p_2}\} + 2 F_{020} |p_2|^2,
\end{aligned}
\tag{3.50}
$$

and $\mathcal{S}_2(\varphi, \psi)$ is the second-order cross terms of φ and ψ, where φ, ψ are column vectors of 3×1.

In addition, by (3.38), we have

$$
F_2^d(\Phi Z \gamma_{n_c}(x)) = \left(\frac{n_c}{\ell} \right)^2 \left(\xi_{n_c}^2(x) - \gamma_{n_c}^2(x) \right) \left(\sum_{m_1+m_2=2} A_{m_1 m_2}^d z_1^{m_1} z_2^{m_2} \right), \tag{3.51}
$$

where

$$
\begin{aligned}
&\xi_{n_c}(x) = \sqrt{\frac{2}{\ell\pi}} \sin\left(\frac{n_c x}{\ell} \right), \\
&A_{20}^d = -2 d_{21} \begin{pmatrix} 0 \\ p_2 p_3 \\ 0 \end{pmatrix} = \overline{A_{02}^d}, \quad A_{11}^d = -4 d_{21} \begin{pmatrix} 0 \\ \mathrm{Re}\{p_2 \overline{p_3}\} \\ 0 \end{pmatrix}.
\end{aligned}
\tag{3.52}
$$

It is easy to verify that

$$
\int_0^{\ell\pi} \gamma_{n_c}^3(x) dx = \int_0^{\ell\pi} \gamma_{n_c}(x) \xi_{n_c}^2(x) dx = 0.
$$

Hence, in terms of (3.48)-(3.51), we have

$$f_2^1(Z,0,0) = \Psi \begin{pmatrix} \left[\widetilde{F}_2(\Phi Z\gamma_{n_c}(x),0), \beta_{n_c}^{(1)}\right] \\ \left[\widetilde{F}_2(\Phi Z\gamma_{n_c}(x),0), \beta_{n_c}^{(2)}\right] \\ \left[\widetilde{F}_2(\Phi Z\gamma_{n_c}(x),0), \beta_{n_c}^{(3)}\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.53}$$

which implies that

$$\mathrm{Proj}_S\left[\left(D_Z f_2^1\right)(Z,0,0)U_2^1(Z,0)\right] = (0,0)^T. \tag{3.54}$$

Step 3: The calculation of $\mathrm{Proj}_S\left(\left(D_W f_2^{(1,1)}\right)(Z,0,0)U_2^2(Z,0)\right)$

Let

$$U_2^2(Z,0) \triangleq h(Z) = \sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x) \in \mathscr{X}^s, \tag{3.55}$$

where

$$h_n(Z) = \sum_{m_1+m_2=2} h_{n,m_1m_2} z_1^{m_1} z_2^{m_2},$$

with

$$h_{n,m_1m_2} = (h_{n,m_1m_2}^{(1)}, h_{n,m_1m_2}^{(2)}, h_{n,m_1m_2}^{(3)})^T.$$

Then, from (3.32), (3.55) and (3.39), we have

$$\begin{aligned} &\left(D_W f_2^{(1,1)}\right)(Z,0,0)\,(h(Z)) \\ =&\Psi \begin{pmatrix} \left[D_W F_2(\Phi Z\gamma_{n_c}(x)+W)|_{W=0}\left(\sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right), \beta_{n_c}^{(1)}\right] \\ \left[D_W F_2(\Phi Z\gamma_{n_c}(x)+W)|_{W=0}\left(\sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right), \beta_{n_c}^{(2)}\right] \\ \left[D_W F_2(\Phi Z\gamma_{n_c}(x)+W)|_{W=0}\left(\sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right), \beta_{n_c}^{(3)}\right] \end{pmatrix}. \end{aligned}$$

By (3.49), we obtain

$$D_W F_2(\Phi Z\gamma_{n_c}(x)+W)|_{w=0}\left(\sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right) = \mathcal{S}_2\left(\Phi Z\gamma_{n_c}(x), \sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right),$$

and

$$\begin{pmatrix} \left[\mathcal{S}_2\left(\Phi Z\gamma_{n_c}(x), \sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right), \beta_{n_c}^{(1)}\right] \\ \left[\mathcal{S}_2\left(\Phi Z\gamma_{n_c}(x), \sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right), \beta_{n_c}^{(2)}\right] \\ \left[\mathcal{S}_2\left(\Phi Z\gamma_{n_c}(x), \sum_{n\in\mathbb{N}_0} h_n(Z)\gamma_n(x)\right), \beta_{n_c}^{(3)}\right] \end{pmatrix} = \sum_{n\in\mathbb{N}_0} b_n\left(\mathcal{S}_2\left(pz_1, h_n(Z)\right) + \mathcal{S}_2\left(\bar{p}z_2, h_n(Z)\right)\right),$$

where

$$b_n = \int_0^{\ell\pi} \gamma_{n_c}^2(x)\gamma_n(x)dx = \begin{cases} \frac{1}{\sqrt{\ell\pi}}, & n=0, \\ \frac{1}{\sqrt{2\ell\pi}}, & n=2n_c, \\ 0, & \text{otherwise.} \end{cases} \tag{3.56}$$

Hence,

$$\mathrm{Proj}_S\left(\left(D_W f_2^{(1,1)}\right)(Z,0,0)U_2^2(Z,0)\right) = \mathcal{H}\left(B_{22}z_1^2 z_2\right), \tag{3.57}$$

where

$$B_{22} = \frac{1}{\sqrt{\ell\pi}}q^T\left(\mathcal{S}_2\left(p, h_{0,11}\right) + \mathcal{S}_2\left(\bar{p}, h_{0,20}\right)\right) + \frac{1}{\sqrt{2\ell\pi}}q^T\left(\mathcal{S}_2\left(p, h_{2n_c,11}\right) + \mathcal{S}_2\left(\bar{p}, h_{2n_c,20}\right)\right).$$

Further, from (3.49), we have

$$\mathcal{S}_2\left(p, h_{n,m_1m_2}\right) = 2F_{200}\left(p_1 h_{n,m_1m_2}^{(1)}\right) + 2F_{110}\left(p_1 h_{n,m_1m_2}^{(2)} + p_2 h_{n,m_1m_2}^{(1)}\right) + 2F_{020}\left(p_2 h_{n,m_1m_2}^{(2)}\right), \tag{3.58}$$

and

$$\mathcal{S}_2(\bar{p}, h_{n,m_1m_2}) = 2F_{200}\left(\overline{p_1} h_{n,m_1m_2}^{(1)}\right) + 2F_{110}\left(\overline{p_1} h_{n,m_1m_2}^{(2)} + \overline{p_2} h_{n,m_1m_2}^{(1)}\right) + 2F_{020}\left(\overline{p_2} h_{n,m_1m_2}^{(2)}\right). \tag{3.59}$$

Thus, we can also obtain that

$$\begin{aligned} B_{22} = \sum_{n=0,2n_c} E_n \Big(&(q_1 f_{20} + q_2 g_{20})(p_1 h_{n,11}^{(1)} + \overline{p_1} h_{n,20}^{(1)}) + (q_1 f_{02} + q_2 g_{02})(p_2 h_{0,11}^{(2)} + \overline{p_2} h_{0,20}^{(2)}) \\ &+ (q_1 f_{11} + q_2 g_{11})(p_1 h_{n,11}^{(2)} + p_2 h_{n,11}^{(1)} + \overline{p_1} h_{n,20}^{(2)} + \overline{p_2} h_{n,20}^{(1)})\Big), \end{aligned} \tag{3.60}$$

where

$$E_n = \begin{cases} \frac{2}{\sqrt{\ell\pi}}, & n=0, \\ \frac{2}{\sqrt{2\ell\pi}}, & n=2n_c. \end{cases}$$

Step 4: The calculation of $\mathbf{Proj_S}\left(\left(\mathbf{D_{W,W_x,W_{xx}} f_2^{(1,2)}}\right)(\mathbf{Z},\mathbf{0})\mathbf{U_2^{(2,d)}}(\mathbf{Z},\mathbf{0})\right)$

From (3.38), we denote

$$\begin{aligned} &F_2^d(\Phi Z\gamma_{n_c}(x), W, W_x, W_{xx}) \triangleq F_2^d(\Phi Z\gamma_{n_c}(x) + W) \\ &= F_2^d(\Phi Z\gamma_{n_c}(x)) + \mathcal{S}_2^{(d,1)}(\Phi Z\gamma_{n_c}(x), W) + \mathcal{S}_2^{(d,2)}(\Phi Z\gamma_{n_c}(x), W_x) + \mathcal{S}_2^{(d,3)}(\Phi Z\gamma_{n_c}(x), W_{xx}) \\ &+ O(|(W, W_x, W_{xx})|^2), \end{aligned} \tag{3.61}$$

with

$$\mathcal{S}_2^{(d,1)}(\Phi Z\gamma_{n_c}(x), W) = 2\left(\frac{n_c}{\ell}\right)^2 d_{21}\gamma_{n_c}(x)\begin{pmatrix} 0 \\ (p_3 z_1 + \overline{p_3} z_2)w^{(2)} \\ 0 \end{pmatrix},$$

$$\mathcal{S}_2^{(d,2)}(\Phi Z\gamma_{n_c}(x), W_x) = 2\left(\frac{n_c}{\ell}\right) d_{21}\xi_{n_c}(x)\begin{pmatrix} 0 \\ (p_3 z_1 + \overline{p_3} z_2)w_x^{(2)} + (p_2 z_1 + \overline{p_2} z_2)w_x^{(3)} \\ 0 \end{pmatrix},$$

$$\mathcal{S}_2^{(d,3)}(\Phi Z\gamma_{n_c}(x), W_{xx}) = -2d_{21}\gamma_{n_c}(x)\begin{pmatrix} 0 \\ (p_2 z_1 + \overline{p_2} z_2)w_{xx}^{(3)} \\ 0 \end{pmatrix}.$$

By (3.55), we denote

$$\begin{aligned} U_{2x}^2(Z,0) &\triangleq h_x(Z) = -\sum_{n\in\mathbb{N}_0}\left(\frac{n}{\ell}\right) h_n(Z)\xi_n(x), \\ U_{2xx}^2(Z,0) &\triangleq h_{xx}(Z) = -\sum_{n\in\mathbb{N}_0}\left(\frac{n}{\ell}\right)^2 h_n(Z)\gamma_n(x). \end{aligned} \tag{3.62}$$

Thus, by (3.37), (3.61) and (3.62), we have

$$\begin{aligned} &D_{W,W_x,W_{xx}} F_2^d(\Phi Z\gamma_{n_c}(x), W, W_x, W_{xx})U_2^{(2,d)}(Z,0) \\ &= \mathcal{S}_2^{(d,1)}(\Phi Z\gamma_{n_c}(x), h(Z)) + \mathcal{S}_2^{(d,2)}(\Phi Z\gamma_{n_c}(x), h_x(Z)) + \mathcal{S}_2^{(d,3)}(\Phi Z\gamma_{n_c}(x), h_{xx}(Z)), \end{aligned} \tag{3.63}$$

and

$$\begin{aligned} &\begin{pmatrix} \left[\mathcal{S}_2^{(d,1)}(\Phi Z\gamma_{n_c}(x), h(Z)), \beta_{n_c}^{(1)}\right] \\ \left[\mathcal{S}_2^{(d,1)}(\Phi Z\gamma_{n_c}(x), h(Z)), \beta_{n_c}^{(2)}\right] \\ \left[\mathcal{S}_2^{(d,1)}(\Phi Z\gamma_{n_c}(x), h(Z)), \beta_{n_c}^{(3)}\right] \end{pmatrix} = 2(n_c/\ell)^2 d_{21}\sum_{n\in\mathbb{N}_0} b_n\left(0, (p_3 z_1 + \overline{p_3} z_2)h_n^{(2)}(Z), 0\right)^T, \\ &\begin{pmatrix} \left[\mathcal{S}_2^{(d,2)}(\Phi Z\gamma_{n_c}(x), h_x(Z)), \beta_{n_c}^{(1)}\right] \\ \left[\mathcal{S}_2^{(d,2)}(\Phi Z\gamma_{n_c}(x), h_x(Z)), \beta_{n_c}^{(2)}\right] \\ \left[\mathcal{S}_2^{(d,2)}(\Phi Z\gamma_{n_c}(x), h_x(Z)), \beta_{n_c}^{(3)}\right] \end{pmatrix} \\ &= -2(n_c/\ell)d_{21}\sum_{n\in\mathbb{N}_0}(n/\ell)c_n\left(0, (p_3 z_1 + \overline{p_3} z_2)h_n^{(2)}(Z) + (p_2 z_1 + \overline{p_2} z_2)h_n^{(3)}(Z), 0\right)^T, \\ &\begin{pmatrix} \left[\mathcal{S}_2^{(d,3)}(\Phi Z\gamma_{n_c}(x), h_{xx}(Z)), \beta_{n_c}^{(1)}\right] \\ \left[\mathcal{S}_2^{(d,3)}(\Phi Z\gamma_{n_c}(x), h_{xx}(Z)), \beta_{n_c}^{(2)}\right] \\ \left[\mathcal{S}_2^{(d,3)}(\Phi Z\gamma_{n_c}(x), h_{xx}(Z)), \beta_{n_c}^{(3)}\right] \end{pmatrix} = 2d_{21}\sum_{n\in\mathbb{N}_0}(n/\ell)^2 b_n\left(0, (p_2 z_1 + \overline{p_2} z_2)h_n^{(3)}(Z), 0\right)^T, \end{aligned} \tag{3.64}$$

where b_n is defined by (3.56) and

$$c_n = \int_0^{\ell\pi} \xi_{n_c}(x)\gamma_{n_c}(x)\xi_n(x)dx = \begin{cases} \dfrac{1}{\sqrt{2\ell\pi}}, & n = 2n_c, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by (3.33), (3.37), (3.55) and (3.62) - (3.64), we obtain

$$\begin{aligned}&\left(D_{W,W_x,W_{xx}} f_2^{(1,2)}\right)(Z,0)U_2^{(2,d)}(Z,0)\\ =&\Psi\begin{pmatrix} \left[D_{W,W_x,W_{xx}} F_2^d(\Phi Z\gamma_{n_c}(x), W, W_x, W_{xx})U_2^{(2,d)}(Z,0), \beta_{n_c}^{(1)}\right] \\ \left[D_{W,W_x,W_{xx}} F_2^d(\Phi Z\gamma_{n_c}(x), W, W_x, W_{xx})U_2^{(2,d)}(Z,0), \beta_{n_c}^{(2)}\right] \\ \left[D_{W,W_x,W_{xx}} F_2^d(\Phi Z\gamma_{n_c}(x), W, W_x, W_{xx})U_2^{(2,d)}(Z,0), \beta_{n_c}^{(3)}\right]\end{pmatrix},\end{aligned}$$

and then we have

$$\mathrm{Proj}_S\left(\left(D_{W,W_x,W_{xx}} f_2^{(1,2)}\right)(Z,0)U_2^{(2,d)}(Z,0)\right) = \mathcal{H}\left(B_{23}z_1^2z_2\right), \tag{3.65}$$

where

$$\begin{aligned}B_{23} =& \frac{2}{\sqrt{\ell\pi}}\left(\frac{n_c}{\ell}\right)^2 d_{21}q_2\left(p_3h_{0,11}^{(2)} + \overline{p_3}h_{0,20}^{(2)}\right)\\ &+ \frac{2}{\sqrt{2\ell\pi}}\left(\frac{n_c}{\ell}\right)^2 d_{21}q_2\left(2\left(p_2h_{2n_c,11}^{(3)} + \overline{p_2}h_{2n_c,20}^{(3)}\right) - \left(p_3h_{2n_c,11}^{(2)} + \overline{p_3}h_{2n_c,20}^{(2)}\right)\right).\end{aligned} \tag{3.66}$$

Summarizing the above calculations, we have the normal form of the Hopf bifurcation at τ_H truncated to the third terms as follows

$$\dot{Z} = BZ + \frac{1}{2!}\begin{pmatrix} B_1z_1\mu \\ \bar{B}_1z_2\mu \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} B_2z_1^2z_2 \\ \bar{B}_2z_1z_2^2 \end{pmatrix} + O\left(|Z|\mu^2 + |Z|^4\right), \tag{3.67}$$

where B_1 is defined by (3.42) and

$$B_2 = B_{21} + \frac{3}{2}(B_{22} + B_{23}), \tag{3.68}$$

with B_{2j}, $j = 1, 2, 3$, determined by (3.47), (3.60) and (3.66), respectively. Through the change of variables $z_1 = w_1 - iw_2$, $z_2 = w_1 + iw_2$ and $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, the normal form (3.67) becomes the form in polar coordinates

$$\dot{\rho} = \kappa_1\mu\rho + \kappa_2\rho^3 + O(\mu^2\rho + |(\mu,\rho)|^4),$$

with

$$\kappa_1 = \frac{1}{2!}\mathrm{Re}\{B_1\}, \quad \kappa_2 = \frac{1}{3!}\mathrm{Re}\{B_2\}. \tag{3.69}$$

For generic mode-n_c Hopf bifurcation, if all other roots of Eq. (2.10) have negative real parts except for a pair of purely imaginary. Then, the direction of the bifurcation and the stability of the nontrivial periodic orbits are determined by the sign of $\kappa_1\kappa_2$ and of κ_2, respectively. The case $\kappa_2 < 0$ is referred to as a supercritical bifurcation, and the case $\kappa_2 > 0$ is referred to as a subcritical bifurcation [28].

The coefficients $h_{0,20}, h_{0,11}, h_{2n_c,20}$, and $h_{2n_c,11}$ in (3.60) and (3.66) are defined by (B.5) and (B.6). See Appendix B for the detailed calculation.

Remark 3.2. The coefficients κ_1 and κ_2 of the normal forms can be determined by the eigenvectors p and q associated with the purely imaginary roots $\pm i\omega_{n_c}$ and the second-order and third-order terms of Taylor expansion of the reaction terms f and g and the diffusion terms. Different from the calculation of normal form of Hopf bifurcation for the standard reaction-diffusion system, the calculation of $\mathrm{Proj}_S\left(\left(\mathrm{D}_{\mathrm{W},\mathrm{W_x},\mathrm{W_{xx}}}\mathrm{f}_2^{(1,2)}\right)(\mathrm{Z},0)\mathrm{U}_2^{(2,\mathrm{d})}(\mathrm{Z},0)\right)$ is main characteristic derived from the nonlinearity of the diffusion terms.

4. Application to the consumer-resource model with type-II functional response and distributed memory

In this section, we consider the consumer-resource model with Holling type-II functional response and study the possible pattern formations induced by the average memory delay τ. For the sake of simplicity, we restrict the spatial domain Ω to be the one-dimensional domain $(0, \ell\pi)$ and choose $\ell = 2$ for the numerical simulations, so we have $\sigma_n = (\frac{n}{\ell})^2$ for $n \in \mathbb{N}$.

The model considered in this section is

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_{11}u_{xx} + u(1-\dfrac{u}{a}) - \dfrac{buv}{1+u}, & 0<x<\ell\pi, t>0, \\ \dfrac{\partial v}{\partial t} = d_{22}v_{xx} - d_{21}(vw_x)_x - cv + \dfrac{buv}{1+u}, & 0<x<\ell\pi, t>0, \\ u_x(0,t) = u_x(\ell\pi,t) = v_x(0,t) = v_x(\ell\pi,t) = 0, & t \ge 0, \end{cases} \tag{4.1}$$

where $w(x,t)$ is defined by (1.2) with $F(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}$ (the weak kernel).

Let $E_* = (u_*, v_*)$ be the positive constant steady state of system (4.1), it is not difficult to obtain that

$$u_* = \frac{c}{b-c}, \quad v_* = \frac{(1+u_*)(a-u_*)}{ab},$$

provided that $b > \frac{c(1+a)}{a}$, and

$$a_{11} = \frac{\zeta(a-1-2\zeta)}{a(1+\zeta)} \begin{cases} <0, \ \frac{a-1}{2} < \zeta < a, \\ \ge 0, \ 0 < \zeta \le \frac{a-1}{2}, \end{cases} \quad a_{12} = -c < 0, \ a_{21} = \frac{a-\zeta}{a(1+\zeta)} > 0, \ a_{22} = 0, \tag{4.2}$$

where $\zeta = \frac{c}{b-c}$.

Obviously, the condition (C_1) is satisfied and it is easy to see that if $\frac{a-1}{2} < \zeta < a$ (equivalent to $a_{11} < 0$), then (C_2) and (C_3) are satisfied since $a_{22} = 0$.

In what follows, we choose

$$a = 2, b = 3.2, c = 1.6, d_{11} = 0.1, d_{22} = 0.2, \ell = 2. \tag{4.3}$$

Then we have $(u_*, v_*) = \left(1, \frac{5}{16}\right)$. It follows from (2.6), (4.2) and (2.9) that

$$d_{21,n}^S = -\left(\frac{n^2}{100} + \frac{16}{5n^2} + \frac{1}{10}\right), n \in \mathbb{N}, \tag{4.4}$$

which are independent of τ and

$$d_{21,4}^S \doteq -0.46 > d_{21,5}^S \doteq -0.4780 > d_{21,3}^S \doteq -0.5456 > d_{21,6}^S \doteq -0.5489 > d_{21,2}^S \doteq -0.94 > \cdots.$$

Thus, we can see that $d_S^* = d_{21,4}^S \doteq -0.46$. At the same time, from (2.6), (4.2) and (2.15), we have

$$\hat{d}_{21}(\sigma_n) = \frac{(1 + \frac{3}{10}n^2)\sqrt{\frac{1}{50}n^4 + \frac{1}{5}n^2 + \frac{32}{5}} + \frac{1}{2}(1 + \frac{3}{10}n^2)^2}{n^2},$$

and then

$$\begin{aligned}\hat{d}_{21}(\sigma_3) \doteq 2.0488 < \hat{d}_{21}(\sigma_2) \doteq 2.1132 < \hat{d}_{21}(\sigma_4) \doteq 2.4420 < \hat{d}_{21}(\sigma_5)\\ \doteq 3.1072 < \hat{d}_{21}(\sigma_6) \doteq 3.9954 < \hat{d}_{21}(\sigma_1) \doteq 4.1898 < \cdots.\end{aligned}$$

In addition, by (2.27) and (2.28), we obtain $\sigma_* \doteq 2.3137$ for the case $d_{11}a_{22} + d_{22}a_{11} < 0$, and then $\tilde{n} = 4$ from (2.30). Therefore, in terms of (2.31), $d_H^* = \min_{1\le n\le 4} \hat{d}_{21}(\sigma_n) = \hat{d}_{21}(\sigma_3) \doteq 2.0488$. From (2.17), we have

$$\tau_2^{\pm} = \frac{\frac{121}{400} - \frac{d_{21}}{2} \pm \sqrt{(\frac{121}{400} - \frac{d_{21}}{2})^2 - \frac{5687}{1000}}}{-517/1000}, \quad \tau_4^{\pm} = \frac{\frac{841}{400} - 2d_{21} \pm \sqrt{(\frac{841}{400} - 2d_{21})^2 - \frac{19343}{125}}}{-667/250},$$
$$\tau_3^{\pm} = \frac{\frac{1369}{1600} - \frac{9}{8}d_{21} \pm \sqrt{(\frac{1369}{1600} - \frac{9}{8}d_{21})^2 - \frac{672179}{320000}}}{-18167/16000}. \tag{4.5}$$

From Theorem 2.11, for $d_{21} \in \left(d_{21,4}^S, \hat{d}_{21}(\sigma_3)\right) \doteq (-0.46, 2.0488)$, the positive constant steady state E_* is locally asymptotically stable for any $\tau \ge 0$. For $d_{21} < d_{21,4}^S \doteq -0.46$, E_* becomes unstable via mode-4 Turing bifurcation. For $d_{21} > \hat{d}_{21}(\sigma_3) \doteq 2.0488$, E_* is locally asymptotically stable for $\tau \in [0, \tau_*) \cup (\tau^*, +\infty)$, and loses its stability via Hopf bifurcation. Fig. 1(a) illustrates these Turing bifurcation lines $d_{21} = d_{21,n}^S$ and Hopf bifurcation curves $\tau = \tau_n^{\pm}$. The dotted region is the stability region. The points are parameter values for numerical simulations and they are: $P_1(2.075, 0.7)$, $P_2(2.075, 1.1)$, $P_3(2.075, 1.32)$, $P_4(2.075, 1.55)$, $P_5(2.075, 2.09)$, $P_6(2.21, 2.17)$, $P_7(-0.47, 1.32)$. Fig. 1(b) is the enlargement of Fig. 1(a) restricted to the region $1.8 < d_{21} < 3.2$, $0 < \tau < 4.2$ and Hopf bifurcation curves $\tau = \tau_2^-$ and $\tau = \tau_3^-$ intersect at the point $P_* \doteq (2.229469, 2.154618)$, which is the double Hopf bifurcation point.

For numerical simulations, we use the equivalent system (3.1) with $f = u(1 - \frac{u}{a}) - \frac{buv}{1+u}$ and $g = -cv + \frac{buv}{1+u}$.

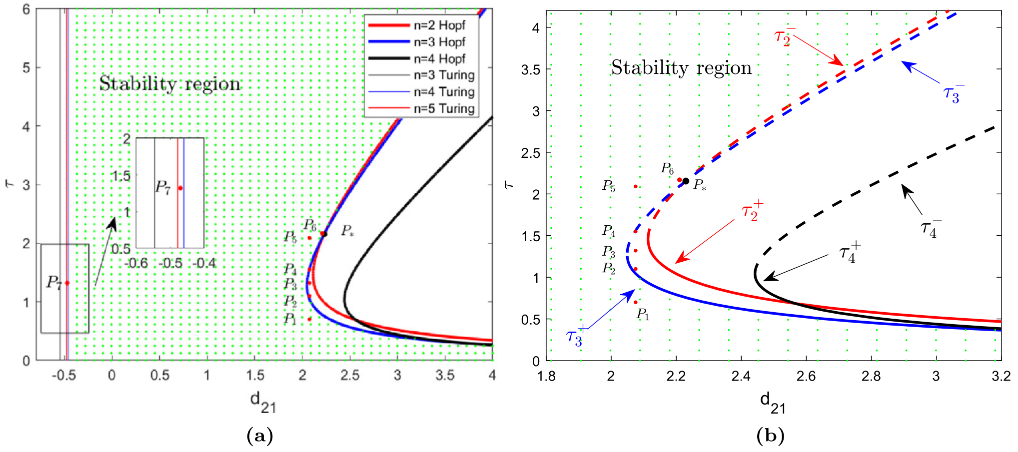


Fig. 1. Stability region and bifurcation diagrams in plane (d_{21}, τ) of system (4.1) with weak kernel delay and parameters in (4.3). The dotted region is the stability region, the Turing bifurcation curves $d_{21} = d^S_{21,n}$ (for $d_{21} < 0$) are potted for $n = 3, 4, 5$ and Hopf bifurcation curves $\tau = \tau^{\pm}_n$ (for $d_{21} > 0$) are potted for $n = 2, 3, 4$. Fig. 1b is the enlargement of Fig. 1a restricted to the region $1.8 < d_{21} < 3.2$, $0 < \tau < 4.2$.

For mode-4 Turing bifurcation $d_{21} = d^S_{21,4} \doteq -0.46$, it follows from Theorem 3.1 that $d'_{21,4}(0) = 0$ and

$$d''_{21,4}(0) \doteq -0.2791 < 0.$$

Thus, system (4.1) undergoes a supercritical pitchfork bifurcation at $d_{21} = d^S_{21,4}$. That is to say that when d_{21} is smaller than and close to $d^S_{21,4}$, system (4.1) has the coexistence of two stable spatially inhomogeneous steady states. Fig. 2 shows the existence of spatially inhomogeneous steady state with the spatial profile $\cos(2x)$-like for point P_7. Figs. 2(a) and 2(c) and Figs. 2(b) and 2(d) have the same parameters but different initial values. Figs. 2(a) and 2(c) is the numerical simulation for the initial value $u(x, 0) = w(x, 0) = 1 + 0.2\cos(2x)$, $v(x, 0) = \frac{5}{16} + 0.2\cos(2x)$, and Figs. 2(b) and 2(d) is the numerical simulation for the initial value $u(x, 0) = w(x, 0) = 1 - 0.2\cos(2x)$, $v(x, 0) = \frac{5}{16} - 0.2\cos(2x)$. Fig. 3 is the projection of $u(x, t)$ shown in Figs. 2(a) and 2(b) for fixed t to $x - u$ plane, which shows the existence of the supercritical pitchfork bifurcation at $d_{21} = d^S_{21,4}$.

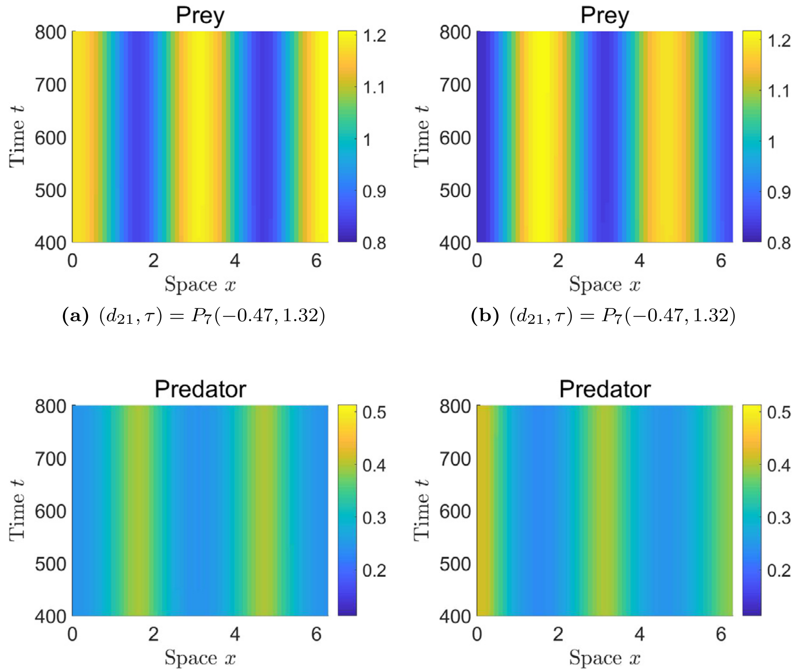
For fixed $d_{21} = 2.075 > d^*_H$, it follows from (4.5) that

$$\tau^+_3 \doteq 1.0439 < \tau^-_3 \doteq 1.5609,$$

System (4.1) undergoes Hopf bifurcations at $\tau = \tau^+_3$ and $\tau = \tau^-_3$ and E_* is asymptotically stable for $\tau \in [0, \tau^+_3) \cup (\tau^-_3, \infty)$. Using the procedure developed in Section 3.2, we have, for $\tau_H = \tau^+_3 \doteq 1.0439$,

$$\kappa_1 \doteq 0.0267 > 0, \ \kappa_2 \doteq -0.3782 < 0,$$

and for $\tau_H = \tau^-_3 \doteq 1.5609$,



(a) $(d_{21}, \tau) = P_7(-0.47, 1.32)$ (b) $(d_{21}, \tau) = P_7(-0.47, 1.32)$

(c) $(d_{21}, \tau) = P_7(-0.47, 1.32)$ (d) $(d_{21}, \tau) = P_7(-0.47, 1.32)$

Fig. 2. Numerical simulations of system (4.1) for (d_{21}, τ) chosen as P_7 in Fig. 1 and different initial values, showing the coexistence of two stable spatially inhomogeneous steady states. (a) and (c): the initial conditions are $u(x,0) = w(x,0) = 1 + 0.2\cos(2x)$, $v(x,0) = \frac{5}{16} + 0.2\cos(2x)$; (b) and (d): the initial conditions are $u(x,0) = w(x,0) = 1 - 0.2\cos(2x)$, $v(x,0) = \frac{5}{16} - 0.2\cos(2x)$.

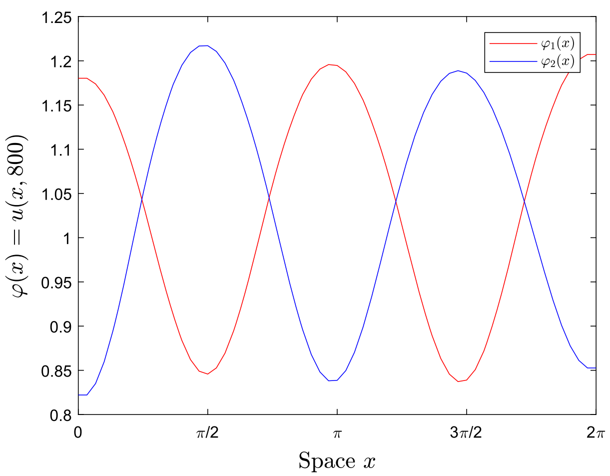
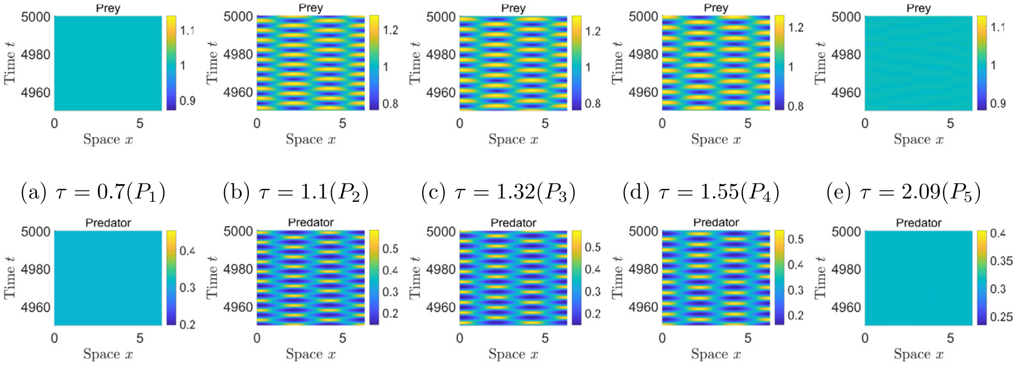


Fig. 3. The projection of $u(x,t)$ shown in Figs. 2(a) and 2(b) for fixed $t = 800$ to $x - u$ plane, showing the existence of the supercritical pitchfork bifurcation at $d_{21} = d^S_{21,4}$.



(a) $\tau = 0.7(P_1)$ (b) $\tau = 1.1(P_2)$ (c) $\tau = 1.32(P_3)$ (d) $\tau = 1.55(P_4)$ (e) $\tau = 2.09(P_5)$

(f) $\tau = 0.7(P_1)$ (g) $\tau = 1.1(P_2)$ (h) $\tau = 1.32(P_3)$ (i) $\tau = 1.55(P_4)$ (j) $\tau = 2.09(P_5)$

Fig. 4. Numerical simulations of system (4.1) for fixed $d_{21} = 2.075$ and for P_j, $j = 1, 2, 3, 4, 5$, shown in Fig. 1. The initial conditions are $u(x, 0) = w(x, 0) = 1 + 0.1\cos(3x/2)$, $v(x, 0) = \frac{5}{16} + 0.1\cos(3x/2)$.

$$\kappa_1 \doteq -0.0165 < 0, \ \kappa_2 \doteq -0.3111 < 0.$$

Therefore, Hopf bifurcations at $\tau = \tau_3^+$ and $\tau = \tau_3^-$ are both supercritical and the corresponding bifurcating periodic solutions are both orbitally asymptotically stable. For fixed $d_{21} = 2.075$ and τ varying from 0.7 to 2.09, Fig. 4 shows the existence and transformation of the spatially inhomogeneous periodic solution with mode-3 spatial patterns for P_j, $j = 1, 2, 3, 4, 5$, shown in Fig. 1. Figs. 4(a) and (f), Figs. 4(e) and (j) show the stability of E_* for $\tau < \tau_3^+$ and $\tau > \tau_3^-$, respectively. Figs. 4(b) and (g) Figs. 4(d) and (i) show the spatially inhomogeneous periodic solutions with the spatial profile $\cos(3x/2)$-like occurring at $\tau < \tau_3^+$ and $\tau > \tau_3^-$, respectively. For $\tau = 1.32$ far away from these two Hopf bifurcation values, Figs. 4(c) and (h) show that the spatially inhomogeneous periodic solution with the spatial profile $\cos(3x/2)$-like still exists. We conjecture that spatially inhomogeneous periodic solution with the spatial profile $\cos(3x/2)$-like always exists for any $\tau \in (\tau_3^+, \tau_3^-)$. Unfortunately, we can not prove this conjecture.

It is worth mentioning that the interacting of mode-2 and mode-3 Hopf bifurcations may produce more complex dynamic behaviors, which urges us to develop and expand the normal form in double Hopf bifurcation to investigate the dynamical classification near the double Hopf bifurcation point P_*. For the point of $P_6(2.21, 2.17)$ in Fig. 1 near the double Hopf bifurcation point P_*, Fig. 5 shows the existence of quasi-periodic spatiotemporal patterns. Fig. 6 illustrates the phase portrait of $u(x, t)$ and $v(x, t)$ for fixed space $x = \frac{\pi}{5}$ in the $u - v$ plane.

5. Concluding remarks

In this paper, we considered the spatiotemporal dynamics of a consumer-resource model with distributed memory. Instead of a discrete delay in [23], the spatial memory was characterized by the distributed delay. The kernel function was chosen as the temporal weak kernel and we investigated the influence of the mean delay on the stability of the positive constant steady state and the induced spatiotemporal dynamics.

For the toxic resources ($d_{21} < 0$), the rate d_{21} of memory-based diffusion can lead to the Turing bifurcation and yield spatially inhomogeneous steady states. There exists a threshold d_S^* of d_{21} such that the positive constant steady state is asymptotically stable for $d_{21} > d_S^*$ and

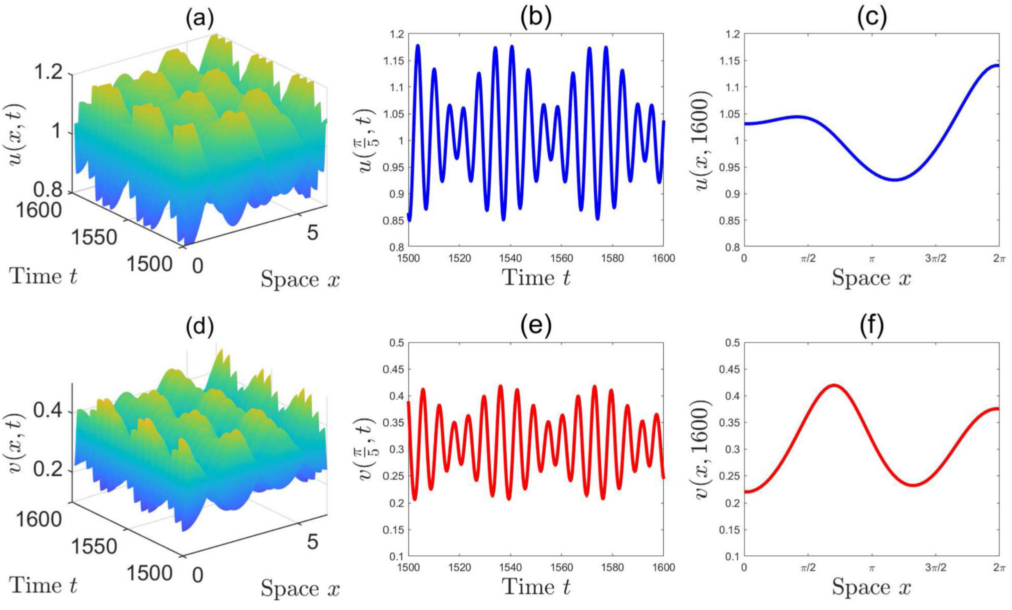


Fig. 5. Numerical simulations of system (4.1) for (d_{21}, τ) being P_6 in Fig. 1 near the double Hopf bifurcation point P_*, showing a quasi-periodic spatiotemporal pattern due to the interaction of mode-2 and mode-3 Hopf bifurcations. (a) and (d): the evolution of spatiotemporal dynamics of the prey u and predator v; (b) and (e): the truncated curves of (a) and (d) for fixed $x = \frac{\pi}{5}$; (c) and (f): the truncated curves of (a) and (d) for fixed $t = 1600$. The initial conditions are $u(x, 0) = w(x, 0) = 1 + 0.1\cos(x)$, $v(x, 0) = \frac{5}{16} + 0.2\cos(3x/2)$.

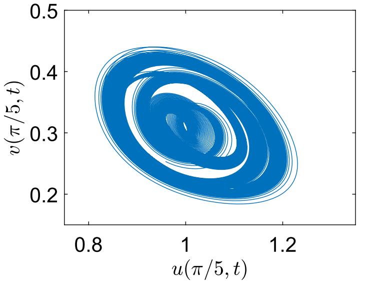


Fig. 6. For fixed space $x = \frac{\pi}{5}$, Figs. 5(b) and 5(e) are plotted in the $u - v$ plane, showing the evolution of the dynamics of the resource u and consumer v in the $u - v$ plane.

unstable for $d_{21} < d_S^*$ and any $\tau \geq 0$, and the system undergoes Turing bifurcation at $d_{21} = d_S^*$. The mean delay τ is independent of the stability and the Turing bifurcation values. We have also derived the formulas to determine the types of the Turing bifurcation and the stability of the bifurcating inhomogeneous steady states. For the consumer-resource model with distributed memory and Holling-II functional response, we found the supercritical pitchfork bifurcation and the coexistence of two spatially inhomogeneous steady states via the Turing bifurcation occurring at $d_{21} = d_S^*$.

For the available resources ($d_{21} > 0$), there exists a threshold d_H^* of d_{21} such that when d_{21} is less than this threshold d_H^*, the positive constant steady state is asymptotically stable for any

$\tau \geq 0$. However, when $d_{21} > d_H^*$, there exist two critical values τ_*, τ^* of the mean delay such that the stability switches occur, i.e., the positive constant steady state is asymptotically stable for $0 \leq \tau < \tau_*$ or $\tau > \tau^*$, and unstable for $\tau \in (\tau_*, \tau^*)$. And the system undergoes spatially inhomogeneous Hopf bifurcation at $\tau = \tau_*$ or $\tau = \tau^*$. We have also derived the normal form of Hopf bifurcation, which determines the direction and stability of the associated Hopf bifurcation. Applying the obtained theoretical results to the consumer-resource model with distributed memory and Holling-II functional response, we found the stable spatially inhomogeneous periodic patterns with different spatial profiles and the quasi-periodic patterns due to the interaction of Hopf bifurcations.

In [23], the influence of the discrete memory delay on the positive constant steady state is investigated for $d_{21} > 0$, where stability switches occur in an appropriate interval (d_{21}^*, d_{21}^{**}) of memory-based diffusion rate, and for $d_{21} > d_{21}^{**}$, if the positive constant steady state loses its stability, then it will never return to stability again. About the role the distributed mean delay and the discrete delay on the dynamics of the resource-consumer model, there are following two main differences

(i) For the case of the distributed mean delay, the positive constant steady state is stable whatever the distributed delay is sufficiently small or large when the memory-based diffusion coefficient d_{21} is larger than some critical value. However, for the case of the discrete delay, the positive constant steady state is always unstable when the discrete delay is small enough or large enough.

(ii) For the case of the distributed mean delay, delay-induced stability switches must occur for any $d_{21} > d_H^*$. However, for the case of the discrete delay, delay-induced stability switches may occur only for the mediate memory-based diffusion rate (i.e., $d_{21}^* < d_{21} < d_{21}^{**}$) and there is no delay-induced stability switches for $d_{21} \geq d_{21}^{**}$.

It seems the distributed memory is more realistic because, from a biological point of view, when the average memory of consumers is old enough, the effect on its movement is negligible, just as there is no effect on it when there is no memory delay. We would like to mention that the explicit memory characterized by the spatiotemporal delay with the weak temporal kernel in the single population model has been considered in [21], which shows that the mean delay does not induce Hopf bifurcation.

Finally, we propose some future topics beyond this study. In this paper, the consumer-resource model with consumer's explicit spatial memory was extended from discrete delay to distributed delay, which is more in line with practical significance. However, only the weak kernel was considered here. When the kernel function is chosen as the general gamma function of order k (the weak kernel corresponds to $k = 0$), how k changes the spatiotemporal dynamics of the system is an open problem. In addition, motivated by [21,25], if the influence of the spatial distribution on the memory is considered, then the spatiotemporal delay should be introduced into the consumer-resource model. The influence of the spatiotemporal delay on the stability of the consumer-resource model with consumer's explicit spatial memory and the corresponding patterns are interesting topics to explore in future.

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors would like to thank the referee for careful reading and helpful suggestions, which greatly improve the presentation of our work. This work was partially supported by grants from National Natural Science Foundation of China (No.11971143) and Zhejiang Provincial Natural Science Foundation of China (No. LZ23A010001) and Natural Sciences and Engineering Research Council of Canada (Discovery Grant RGPIN-2020-03911 and Accelerator Grant RGPAS-2020-00090).

Appendix A. Proof of Theorem 3.1

The steady state of Eq. (3.1) is a solution of the following system:

$$\begin{cases} d_{11}\Delta u + f(u,v) = 0, & x \in \Omega, \\ d_{22}\Delta v - d_{21} div(v\nabla w) + g(u,v) = 0, & x \in \Omega, \\ \dfrac{1}{\tau}(u - w) = 0, & x \in \Omega, \\ \partial_{\boldsymbol{n}} u = \partial_{\boldsymbol{n}} v = \partial_{\boldsymbol{n}} w = 0, & x \in \partial\Omega. \end{cases} \tag{A.1}$$

According to Theorem 1.7 in [4], we state our proof as follows. From (A.1) and taking d_{21} as a parameter, we construct the nonlinear mapping $F : \mathbb{R}^- \times X^3 \to Y^3$ as

$$F(d_{21}, u, v, w) = \begin{pmatrix} d_{11}\Delta u + f(u,v) \\ d_{22}\Delta v - d_{21} div(v\nabla w) + g(u,v) \\ \frac{1}{\tau}(u-w) \end{pmatrix}, \tag{A.2}$$

where X^3, Y^3 are Banach spaces. As we can see that $F(d_{21}, u_*, v_*, u_*) = 0$ for any $d_{21} < 0$, and the Fréchet derivative of F with respect to (u, v, w) is

$$F_{(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[\varphi, \psi, \vartheta] = \begin{pmatrix} d_{11}\Delta\varphi + a_{11}\varphi + a_{12}\psi \\ d_{22}\Delta\psi - d^S_{21,n} v_* \Delta\vartheta + a_{21}\varphi + a_{22}\psi \\ \frac{1}{\tau}(\varphi - \vartheta) \end{pmatrix} := L[\varphi, \psi, \vartheta]. \tag{A.3}$$

Step1: We are supposed to investigate the null space of L, denote it by $\mathcal{N}(L)$. We define Matrix J^S_n by

$$J^S_n = \begin{pmatrix} -d_{11}\sigma_n + a_{11} & a_{12} & 0 \\ a_{21} & -d_{22}\sigma_n + a_{22} & d_{21} v_* \sigma_n \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} \end{pmatrix}. \tag{A.4}$$

Then from previous statement in Section 2, we know that $R_n(d^S_{21,n}) = 0$ and $n \in \mathbb{N}$, which implies that $\lambda = 0$ is an eigenvalue of (A.4). Note that $P_n, Q_n > 0$, therefore $\lambda = 0$ is a single root of Eq. (2.10), again with the fact that σ_n is also the single eigenvalue of (2.3) and $d^S_{21,n} \neq d^S_{21,k}$ for any $k \in \mathbb{N}$, $k \neq n$. We say that $\lambda = 0$ is the single root of linear operator L, and $\mathcal{N}(L) = span\{q = (1, h_n, 1)^T \gamma_n(x)\}$, since

$$Lq = J_n^S q = J_n^S \begin{pmatrix} 1 \\ h_n \\ 1 \end{pmatrix} \gamma_n(x) = 0,$$

where $h_n = \frac{d_{11}\sigma_n - a_{11}}{a_{12}}$. Thus, $\dim(\mathcal{N}(L)) = 1$.

Step2: In this part, we are concentrated on calculating the range space of L, denote it by $\mathcal{R}(L)$. Clearly, $\mathcal{R}(L)$ is given by $\{(f_1, f_2, f_3)^T \in Y^3 | \langle q^*, (f_1, f_2, f_3)^T \rangle = 0\}$, where $q^* \in \mathcal{N}(L^*)$, L^* is the adjoint operator of L and $\langle \cdot, \cdot \rangle$ is the scalar product of two complex vectors defined by

$$\langle \psi, \varphi \rangle = \psi^T \varphi, \quad \text{for} \quad \varphi, \psi \in \mathbb{C}^3. \tag{A.5}$$

By simple calculations, we have

$$L^*[\varphi, \psi, \vartheta] = \begin{pmatrix} d_{11}\Delta\varphi + a_{11}\varphi + a_{21}\psi + \frac{1}{\tau}\vartheta \\ a_{12}\varphi + d_{22}\Delta\psi + +a_{22}\psi \\ -d_{21,n}^S v_* \Delta\psi - \frac{1}{\tau}\vartheta \end{pmatrix},$$

which means that

$$\mathcal{N}(L^*) = span\left\{ q^* = (1, r_n, z_n)^T \gamma_n(x) \right\},$$

with

$$r_n = \frac{a_{12}}{d_{22}\sigma_n - a_{22}}, \quad z_n = \frac{\tau J_n}{d_{22}\sigma_n - a_{22}}.$$

Hence we obtain

$$\mathcal{R}(L) = \{(f_1, f_2, f_3) \in Y^3 | \int_\Omega (f_1 + r_n f_2 + z_n f_3)\gamma_n(x)dx = 0\}, \tag{A.6}$$

and co-dim$(\mathcal{R}(L)) = 1$.

Step3: Our purpose is to show that $F_{d_{21}(u,v,w)}(d_{21,n}^S, u_*, v_*, u_*)[q] \notin \mathcal{R}(L)$ in this step. Obviously, we can get

$$F_{d_{21}(u,v,w)}(d_{21,n}^S, u_*, v_*, u_*)[q] = (0, v_*\sigma_n\gamma_n(x), 0)^T, \tag{A.7}$$

from (A.3), and notice

$$\int_\Omega (0 + r_n v_* \sigma_n \gamma_n + 0)\gamma_n dx = r_n v_* \sigma_n \int_\Omega \gamma_n^2(x)dx \neq 0,$$

thus according to (A.6), we have

$$F_{d_{21}(u,v,w)}(d_{21,n}^S, u_*, v_*, u_*)[q] \notin \mathcal{R}(L).$$

From **Step1-3**, we can now apply the Theorem 1.7 of [4] and obtain the part (i) in Th. 3.1. We would show the part (ii) in the following steps.

Step4: At present, letting $\Omega = (0, \ell\pi)$, thus $\gamma_n(x) = \cos(\frac{nx}{\ell})$, $\sigma_n = \left(\frac{n}{\ell}\right)^2$. We consider the Turing bifurcation direction and its stability in Γ_n. In this case, we have $q = (1, h_n, 1)^T \cos(\frac{nx}{\ell})$. From [17], we have

$$d'_{21,n}(0) = -\frac{\left\langle \ell, F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q, q]\right\rangle}{2\left\langle \ell, F_{d_{21}(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q]\right\rangle}, \tag{A.8}$$

where $\ell \in Y$ satisfies $\mathcal{N}(\ell) = \mathcal{R}(L)$ and can be calculated as

$$\left\langle \ell, (f_1, f_2, f_3)^T \right\rangle = \int_0^{\ell\pi} (f_1 + r_n f_2 + z_n f_3) \cos\left(\frac{nx}{\ell}\right) dx.$$

As a result, and using (A.7), we obtain

$$\left\langle \ell, F_{d_{21}(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q]\right\rangle = r_n v_* \sigma_n \int_0^{\ell\pi} \cos^2\left(\frac{nx}{\ell}\right) dx = \frac{r_n v_* \sigma_n \ell\pi}{2}.$$

Again, from (A.2), we obtain the 2-order Fréchet derivative shown as

$$F_{(u,v,w)(u,v,w)}(d_{21}, u, v, w)[\varphi, \psi, \vartheta][\varphi, \psi, \vartheta] = \begin{pmatrix} f''_{20}(u,v)\varphi^2 + 2f''_{11}(u,v)\varphi\psi + f''_{02}(u,v)\psi^2 \\ -2d_{21}\psi'\vartheta' - 2d_{21}\psi\vartheta'' + g''_{20}(u,v)\varphi^2 + 2g''_{11}(u,v)\varphi\psi + g''_{02}(u,v)\psi^2 \\ 0 \end{pmatrix}. \tag{A.9}$$

Hence, we have

$$F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][q] = \begin{pmatrix} f''\left(1 + \cos(\frac{2nx}{\ell})\right) \\ 2d^S_{21,n}\sigma_n h_n \cos(\frac{2nx}{\ell}) + g''\left(1 + \cos(\frac{2nx}{\ell})\right) \\ 0 \end{pmatrix}, \tag{A.10}$$

where f'', g'' are defined in (3.7). Thus

$$\begin{aligned} &\left\langle \ell, F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q, q]\right\rangle \\ &= \int_0^{\ell\pi} \left[f''\left(1 + \cos\left(\frac{2nx}{\ell}\right)\right)\cos\left(\frac{nx}{\ell}\right) + r_n\left(2d^S_{21,n}\sigma_n h_n \cos\left(\frac{2nx}{\ell}\right) + g''\left(1 + \cos\left(\frac{2nx}{\ell}\right)\right)\right)\cos\left(\frac{nx}{\ell}\right)\right] dx = 0, \end{aligned}$$

this, together with (A.8), we obtain that $d'_{21,n}(0) = 0$.

Next, we show the calculation of $d''_{21,n}(0)$ which determine the bifurcation direction from [17] and the expression of $d''_{21,n}(0)$ is given by

$$d''_{21,n}(0) = -\frac{\left\langle \ell, F_{(u,v,w)(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][q][q]\right\rangle + 3\left\langle \ell, F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][\Theta]\right\rangle}{3\left\langle \ell, F_{d_{21}(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q]\right\rangle},$$

where $\Theta = (\Theta_1, \Theta_2, \Theta_3)$ is the unique solution of

$$F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][q] + F_{(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[\Theta] = 0. \tag{A.11}$$

Further, from (A.2), we obtain

$$\begin{aligned} &F_{(u,v,w)(u,v,w)(u,v,w)}(d_{21}, u, v, w)[\varphi, \psi, \vartheta][\varphi, \psi, \vartheta][\varphi, \psi, \vartheta] \\ &= \begin{pmatrix} f'''_{30}(u,v)\varphi^3 + 3f'''_{21}(u,v)\varphi^2\psi + 3f'''_{12}(u,v)\varphi\psi^2 + f'''_{03}(u,v)\psi^3 \\ g'''_{30}(u,v)\varphi^3 + 3g'''_{21}(u,v)\varphi^2\psi + 3g'''_{12}(u,v)\varphi\psi^2 + g'''_{03}(u,v)\psi^3 \\ 0 \end{pmatrix}, \end{aligned}$$

thus,

$$\left\langle \ell, F_{(u,v,w)(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][q][q]\right\rangle = \frac{3\ell\pi}{4}\left(f''' + r_n g'''\right),$$

where f''' and g''' are defined by (3.6).

At what follows, we show the calculation of $\left\langle \ell, F_{(u,v,w)(u,v,w)}(d^S_{2,n}, u_*, v_*, u_*)[q][\Theta]\right\rangle$.

By (A.10) and (A.11), we could assume $\Theta = (\Theta_1, \Theta_2, \Theta_3)^T$ has the form as follows:

$$\Theta_1 = \Theta_1^0 + \Theta_1^2 \cos\left(\frac{2nx}{\ell}\right), \Theta_2 = \Theta_2^0 + \Theta_2^2 \cos\left(\frac{2nx}{\ell}\right), \Theta_3 = \Theta_3^0 + \Theta_3^2 \cos\left(\frac{2nx}{\ell}\right), \tag{A.12}$$

since $F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][q]$ consists of only constant and $\cos\left(\frac{2nx}{\ell}\right)$ terms by (A.10). Now, noticing (A.3) and substituting (A.12) into (A.11), we obtain that

$$\begin{aligned} &\begin{pmatrix} f''\left(1+\cos(\frac{2nx}{\ell})\right) \\ 2d^S_{21,n}\sigma_n h_n \cos(\frac{2nx}{\ell}) + g''\left(1+\cos(\frac{2nx}{\ell})\right) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -4d_{11}\sigma_n\Theta_1^2\cos(\frac{2nx}{\ell}) + a_{11}\left(\Theta_1^0 + \Theta_1^2\cos(\frac{2nx}{\ell})\right) + a_{12}\left(\Theta_2^0 + \Theta_2^2\cos(\frac{2nx}{\ell})\right) \\ -4d_{22}\sigma_n\Theta_2^2\cos(\frac{2nx}{\ell}) + 4d^S_{21,n}\sigma_n v_*\Theta_3^2\cos(\frac{2nx}{\ell}) + a_{21}\left(\Theta_1^0 + \Theta_1^2\cos(\frac{2nx}{\ell})\right) + a_{22}\left(\Theta_2^0 + \Theta_2^2\cos(\frac{2nx}{\ell})\right) \\ \frac{1}{\tau}\left(\Theta_1^0 - \Theta_3^0 + (\Theta_1^2 - \Theta_3^2)\cos(\frac{2nx}{\ell})\right) \end{pmatrix}. \end{aligned}$$

Matching the coefficients with respect to the constant and $\cos\left(\frac{2nx}{\ell}\right)$ and obtaining that $\Theta_1^0 = \Theta_3^0$, $\Theta_1^2 = \Theta_3^2$ and Θ_1^0, Θ_2^0, Θ_1^2, Θ_2^2, which are given by (3.5). Thus, by (A.9), we have

$$\begin{aligned} &F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][\Theta] \\ &= \begin{pmatrix} (f''_{20} + f''_{11}h_n)\cos(\frac{nx}{\ell})\left(\Theta_1^0 + \Theta_1^2\cos(\frac{2nx}{\ell})\right) + (f''_{11} + f''_{02}h_n)\cos(\frac{nx}{\ell})\left(\Theta_2^0 + \Theta_2^2\cos(\frac{2nx}{l})\right) \\ d^S_{21,n}\sigma_n\left((4h_n\Theta_1^2 + \Theta_2^2)\cos(\frac{nx}{\ell})\cos(\frac{2nx}{\ell}) - (2h_n + 2\Theta_2^2)\sin(\frac{nx}{\ell})\sin(\frac{2nx}{\ell}) + \Theta_2^0\cos(\frac{nx}{\ell})\right) + \widehat{g} \\ 0 \end{pmatrix}, \end{aligned}$$

where

$$\widehat{g} = (g''_{20} + g''_{11} h_n) \cos\left(\frac{nx}{\ell}\right)\left(\Theta_1^0 + \Theta_1^2 \cos\left(\frac{2nx}{\ell}\right)\right)$$
$$+ (g''_{11} + g''_{02} h_n) \cos\left(\frac{nx}{\ell}\right)\left(\Theta_2^0 + \Theta_2^2 \cos\left(\frac{2nx}{\ell}\right)\right).$$

So we obtain that

$$\begin{aligned}
&\left\langle \ell, F_{(u,v,w)(u,v,w)}(d^S_{21,n}, u_*, v_*, u_*)[q][\Theta] \right\rangle \\
&= \left((f''_{20} + r_n g''_{20})\Theta_1^0 + (f''_{11} + r_n g''_{11})(\Theta_2^0 + h_n \Theta_1^0) + (f''_{02} + r_n g''_{02}) h_n \Theta_2^0 + r_n d^S_{21,n} \sigma_n \Theta_2^0 \right) \int_0^{\ell\pi} \cos^2\left(\frac{nx}{\ell}\right) dx \\
&+ \left((f''_{20} + r_n g''_{20})\Theta_1^2 + (f''_{11} + r_n g''_{11})(\Theta_2^2 + h_n \Theta_1^2) + (f''_{02} + r_n g''_{02}) h_n \Theta_2^2 + r_n d^S_{21,n} \sigma_n (4 h_n \Theta_1^2 + \Theta_2^2) \right) \int_0^{\ell\pi} \cos^2\left(\frac{nx}{\ell}\right) \cos\left(\frac{2nx}{\ell}\right) dx \\
&- 2 r_n d^S_{21,n} \sigma_n (h_n + \Theta_2^2) \int_0^{\ell\pi} \sin\left(\frac{nx}{\ell}\right) \sin\left(\frac{2nx}{\ell}\right) \cos\left(\frac{nx}{\ell}\right) dx \\
&= \frac{\ell\pi}{4} \left((f''_{20} + r_n g''_{20})(2\Theta_1^0 + \Theta_1^2) + (f''_{11} + r_n g''_{11}) \left(2\Theta_2^0 + \Theta_2^2 + h_n (2\Theta_1^0 + \Theta_1^2)\right) + (f''_{02} + r_n g''_{02}) h_n (2\Theta_2^0 + \Theta_2^2) \right) \\
&+ \frac{\ell\pi}{4} \left(r_n d^s_{21,n} \sigma_n (2\Theta_2^0 + 4 h_n \Theta_1^2 - \Theta_2^2 - 2h_n) \right).
\end{aligned}$$

Hence, we can obtain the expression of $d''_{21,n}(0)$ shown in (3.4).

Step5: In this part, we show the stability of the bifurcating non-constant steady states, which can be determined by the sign of $\lambda(s)$, satisfying

$$\lim_{s \to 0} \frac{-s d'_{21,n}(s) m'\left(d^S_{21,n}\right)}{\lambda(s)} = 1, \tag{A.13}$$

where $m(d_{21})$ and $\lambda(s)$ are the eigenvalues of

$$F_{(u,v,w)}(d_{21}, u_*, v_*, u_*)[\varphi(d_{21}), \psi(d_{21}), \vartheta(d_{21})]$$
$$= m(d_{21}) K[\varphi(d_{21}), \psi(d_{21}), \vartheta(d_{21})], \ d_{21} \in \left(d^S_{21,n} - \varepsilon, d^S_{21,n} + \varepsilon\right),$$

and

$$F_{(u,v,w)}(d_{21,n}(s), U_n(s), V_n(s), W_n(s))[\Upsilon(s), \Phi(s), \Psi(s)] = \lambda(s) K[\Upsilon(s), \Phi(s), \Psi(s)], \quad s \in (-\delta, \delta),$$

with $K : X \to Y$ is the inclusion map satisfying $K(u) = u$, $m\left(d^S_{2,n}\right) = \lambda(0) = 0$ and

$$\left(\varphi\left(d^S_{21,n}\right), \psi\left(d^S_{21,n}\right), \vartheta\left(d^S_{21,n}\right)\right) = (\Upsilon(0), \Phi(0), \Psi(0)) = (1, h_n, 1) \cos\left(\frac{nx}{\ell}\right).$$

Now, we analyze the stability of bifurcation direction of steady-states. Let $d_{21} = d^S_{21,N} = d^*_S$, according to Lemma 2.3 and Theorem 2.11, the constant equilibrium (u_*, v_*, u_*) is stable and $m(d_{21}) < 0$ when $d_{21} > d^S_{21,N}$ and it is unstable and $m(d_{21}) < 0$ when $d_{21} < d^S_{21,N}$.

One can calculate the $m'(d_{21})$ by taking derivative of Eq. (2.10) with respect to d_{21} and treating λ as the function of d_{21}, as a result, we obtain

$$m'(d_{21}) = \frac{v_* a_{12}\sigma_n}{\tau(3\lambda^2(d_{21}) + 2\lambda(d_{21})P_n + Q_n)},$$

which implies that

$$m'(d^S_{21,N}) = \frac{v_* a_{12}\sigma_n}{\tau Q_n} < 0,$$

since $a_{12} < 0$, $Q_n > 0$. Moreover, if $d''_{21,N}(0) < 0$, then

$$d'_{21,N}(s) \begin{cases} > 0, & s \in (-\delta, 0), \\ < 0, & s \in (0, \delta). \end{cases}$$

Hence $-sd'_{21,N}(s)m'(d^S_{21,N}) < 0$ for $s \in (-\delta, \delta)\backslash\{0\}$, and consequently $\lambda(s) < 0$ by (A.13) and the bifurcating steady state solutions are locally asymptotically stable. Similarly, if $d''_{21,N}(0) > 0$, the bifurcating solutions are unstable. For any other bifurcation at $d_{21} = d^S_{21,n} < d^*_S$, the constant solution (u_*, v_*, u_*) is already unstable, therefore all bifurcating solutions are also unstable. Hence, we complete the proof of this Theorem.

Appendix B. The calculations of $\mathbf{h_{0,20}}$, $\mathbf{h_{0,11}}$, $\mathbf{h_{2n_c,20}}$, and $\mathbf{h_{2n_c,11}}$

It follows from (3.29) that

$$M_2^2(h_n(Z)\gamma_n(x)) = D_Z\left(h_n(Z)\gamma_n(x)\right)BZ - \mathscr{L}\left(h_n(Z)\gamma_n(x)\right),$$

which leads to

$$\begin{pmatrix} \left[M_2^2(h_n(Z)\gamma_n(x)), \beta_n^{(1)}\right] \\ \left[M_2^2(h_n(Z)\gamma_n(x)), \beta_n^{(2)}\right] \\ \left[M_2^2(h_n(Z)\gamma_n(x)), \beta_n^{(3)}\right] \end{pmatrix} = 2i\omega_{n_c}\left(h_{n,20}z_1^2 - h_{n,02}z_2^2\right) + \left(\frac{n}{\ell}\right)^2 d_0 h_n(Z) - L_0(h_n(Z)). \tag{B.1}$$

In addition, by (3.19) and (3.26), we get

$$f_2^2(Z,0,0) = \widetilde{F}_2(\Phi Z\gamma_{n_c}(x), 0) - \Phi\left\langle \Psi, \begin{pmatrix} \left[\widetilde{F}_2(\Phi Z\gamma_{n_c}(x), 0), \beta_{n_c}^{(1)}\right] \\ \left[\widetilde{F}_2(\Phi Z\gamma_{n_c}(x), 0), \beta_{n_c}^{(2)}\right] \\ \left[\widetilde{F}_2(\Phi Z\gamma_{n_c}(x), 0), \beta_{n_c}^{(3)}\right] \end{pmatrix}\right\rangle \gamma_{n_c}(x). \tag{B.2}$$

Then, from (3.48)-(3.52) and note the fact that

$$\int_0^{\ell\pi} \gamma_{n_c}(x)\gamma_0(x)dx = \int_0^{\ell\pi} \gamma_{n_c}(x)\gamma_{2n_c}(x)dx = 0,$$

we have

$$\begin{pmatrix} \left[f_2^2(Z,0,0), \beta_n^{(1)}\right] \\ \left[f_2^2(Z,0,0), \beta_n^{(2)}\right] \\ \left[f_2^2(Z,0,0), \beta_n^{(3)}\right] \end{pmatrix} = \begin{cases} \frac{1}{\sqrt{\ell\pi}} \sum\limits_{m_1+m_2=2} A_{m_1m_2} z_1^{m_1} z_2^{m_2}, & n=0, \\ \frac{1}{\sqrt{2\ell\pi}} \sum\limits_{m_1+m_2=2} \widetilde{A}_{m_1m_2} z_1^{m_1} z_2^{m_2}, & n=2n_c, \end{cases} \tag{B.3}$$

where

$$\widetilde{A}_{m_1m_2} = A_{m_1m_2} - 2(n_c/\ell)^2 A^d_{m_1m_2}, \ m_1, m_2 \in \mathbb{N}_0, m_1+m_2=2,$$

and $A_{m_1m_2}$, $A^d_{m_1m_2}$ are defined by (3.50), (3.52), respectively.

From (3.37), (B.1)-(B.3) and matching the coefficients of z_1^2 and z_1z_2, we obtain

$$\begin{cases} z_1^2 : 2i\omega_{n_c} h_{n,20} + \left(\frac{n}{\ell}\right)^2 d_0 h_{n,20} - L_0(h_{n,20}) = \begin{cases} \frac{1}{\sqrt{\ell\pi}} A_{20}, & n=0, \\ \frac{1}{\sqrt{2\ell\pi}} \widetilde{A}_{20}, & n=2n_c, \end{cases} \\ z_1z_2 : \left(\frac{n}{\ell}\right)^2 d_0 h_{n,11} - L_0(h_{n,11}) = \begin{cases} \frac{1}{\sqrt{\ell\pi}} A_{11}, & n=0, \\ \frac{1}{\sqrt{2\ell\pi}} \widetilde{A}_{11}, & n=2n_c. \end{cases} \end{cases} \tag{B.4}$$

Solving these equations yields

$$\begin{cases} h_{0,20} = \frac{1}{\sqrt{\ell\pi}} \left(2i\omega_{n_c} I_3 - \mathcal{M}_0\right)^{-1} A_{20}, \\ h_{2n_c,20} = \frac{1}{\sqrt{2\ell\pi}} \left(2i\omega_{n_c} I_3 - \mathcal{M}_{2n_c}\right)^{-1} \widetilde{A}_{20}, \end{cases} \tag{B.5}$$

and

$$\begin{cases} h_{0,11} = -\frac{1}{\sqrt{\ell\pi}} \left(\mathcal{M}_0\right)^{-1} A_{11}, \\ h_{2n_c,11} = -\frac{1}{\sqrt{2\ell\pi}} \left(\mathcal{M}_{2n_c}\right)^{-1} \widetilde{A}_{11}, \end{cases} \tag{B.6}$$

where the matrix $\mathcal{M}_n$ is defined by (3.17).

References

[1] Q. An, C. Wang, H. Wang, Analysis of a spatial memory model with nonlocal maturation delay and hostile boundary condition, Discrete Contin. Dyn. Syst. 40 (10) (2020) 5845–5868.

[2] E. Beretta, D. Breda, Discrete or distributed delay? Effects on stability of population growth, Math. Biosci. Eng. 13 (1) (2016) 19–41.

[3] K.L. Cooke, Z. Grossman, Discrete delay, distributed delay and stability switches, J. Math. Anal. Appl. 86 (2) (1982) 592–627.
[4] M.G. Crandall, P.H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. 8 (2) (1971) 321–340.
[5] W.F. Fagan, E. Gurarie, S. Bewick, A. Howard, R.S. Cantrell, C. Cosner, Perceptual ranges, information gathering, and foraging success in dynamic landscapes, Am. Nat. 189 (5) (2017) 474–489.
[6] T. Faria, Normal forms and Hopf bifurcation for partial differential equations with delays, Trans. Am. Math. Soc. 352 (2000) 2217–2238.
[7] T. Faria, L.T. Magalhaes, Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation, J. Differ. Equ. 122 (2) (1995) 181–200.
[8] S. Gourley, S. Ruan, Dynamics of the diffusive Nicholson's blowflies equation with distributed delay, Proc. R. Soc. Edinb., Sect. A, Math. 130 (6) (2000) 1275–1291.
[9] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge Univ. Press, Cambridge, 1981.
[10] M. Lewis, J. Murray, Modelling territoriality and wolf-deer interactions, Nature 366 (6457) (1993) 738–740.
[11] S. Li, Z. Li, B. Dai, Stability and Hopf bifurcation in a prey-predator model with memory-based diffusion, Discrete Contin. Dyn. Syst., Ser. B (2022), https://doi.org/10.3934/dcdsb.2022025.
[12] B. Liu, H.J. Marquez, Uniform stability of discrete delay systems and synchronization of discrete delay dynamical networks via Razumikhin technique, IEEE Trans. Circuits Syst. I, Regul. Pap. 55 (9) (2008) 2795–2805.
[13] P.R. Moorcroft, M.A. Lewis, R.L. Crabtree, Home range analysis using a mechanistic home range model, Ecology 80 (5) (1999) 1656–1665.
[14] J.R. Potts, M.A. Lewis, How memory of direct animal interactions can lead to territorial pattern formation, J. R. Soc. Interface 13 (118) (2016) 20160059.
[15] S. Ruan, G.S.K. Wolkowicz, Bifurcation analysis of a chemostat model with a distributed delay, J. Math. Anal. Appl. 204 (1996) 786–812.
[16] U.E. Schläegel, M.A. Lewis, Detecting effects of spatial memory and dynamic information on animal movement decisions, Methods Ecol. Evol. 5 (11) (2014) 1236–1246.
[17] J. Shi, Persistence and bifurcation of degenerate solutions, J. Funct. Anal. 169 (2) (1999) 494–531.
[18] J. Shi, C. Wang, H. Wang, Diffusive spatial movement with memory and maturation delays, Nonlinearity 32 (9) (2019) 3188–3208.
[19] J. Shi, C. Wang, H. Wang, Spatial movement with diffusion and memory-based self-diffusion and cross-diffusion, J. Differ. Equ. 305 (2021) 242–269.
[20] J. Shi, C. Wang, H. Wang, X. Yan, Diffusive spatial movement with memory, J. Dyn. Differ. Equ. 32 (1) (2020) 979–1002.
[21] Q. Shi, J. Shi, H. Wang, Spatial movement with distributed memory, J. Math. Biol. 82 (4) (2021) 1–32.
[22] Y. Song, Y. Peng, T. Zhang, The spatially inhomogeneous Hopf bifurcation induced by memory delay in a memory-based diffusion system, J. Differ. Equ. 300 (2021) 597–624.
[23] Y. Song, J. Shi, H. Wang, Spatiotemporal dynamics of a diffusive consumer-resource model with explicit spatial memory, Stud. Appl. Math. 148 (1) (2021) 373–395.
[24] Y. Song, S. Wu, H. Wang, Spatiotemporal dynamics in the single population model with memory-based diffusion and nonlocal effect, J. Differ. Equ. 267 (11) (2019) 6316–6351.
[25] Y. Song, S. Wu, H. Wang, Memory-based movement with spatiotemporal distributed delays in diffusion and reaction, Appl. Math. Comput. 404 (2021) 126254.
[26] Y. Song, T. Zhang, Y. Peng, Turing-Hopf bifurcation in the reaction-diffusion equations and its applications, Commun. Nonlinear Sci. Numer. Simul. 33 (2016) 229–258.
[27] H. Wang, Y. Salmaniw, PDE guidance for cognitive animal movement, arXiv preprint, arXiv:2201.09150.
[28] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, New York, 2003.