General coefficients

MATH 334

Dept of Mathematical and Statistical Sciences
University of Alberta

General coefficients (possibly nonpolynomial)

For simplicity, we deal with the homogeneous case

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0,$$

but now P, Q, and R need not be polynomial. In standard form, this becomes

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where $p = Q/P$, $q = R/P$. We need a more general definition of ordinary and singular points.

Definition (Ordinary and singular points)

If p and q are *analytic functions* at $x = x_0$, then x_0 is an *ordinary point* of the differential equations above. Otherwise, it is a *singular point*.

Analytic functions

- Recall an *analytic function* $f(x)$ at x_0 is one that equals its Taylor series $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for all x in an open interval about x_0.
- Need this so that we can replace coefficients in DE by their Taylor series.
- Compare to previous definition:
  - Previously, P, Q, and R were polynomials, so $p = Q/P$ and $q = R/P$ were *rational functions* (ratios of polynomials).
  - Rational functions fail to be analytic only where they fail to be defined, which is where P is zero and Q and R are not.
  - Previous definition valid for polynomials: singular points occurred if $P(x_0) = 0$ after common factors were removed from P, Q, R.
- This definition includes our former definition as the special case where P, Q, and R are polynomials.

Examples

- $x^2y'' + xy' + x^3y = 0$

- $x^2y'' + xy' + e^xy = 0$

Existence of solutions

Theorem

Say that x_0 is an ordinary point of $P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$, so that $p = Q/P$ and $q = R/P$ are analytic functions at x_0. Then

(i) *the general solution can be written as*

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x).$$

The functions y_1 and y_2 are analytic functions at x_0 and form a fundamental set of solutions of the equation. The constants a_0, a_1 are arbitrary.

(ii) *The radius of convergence of the solution $y(x)$ is at least as great as the distance between x_0 and the closest singular point. Note that all singular points,* **including the complex ones**, *must be taken into account in the computation of the radius of convergence. If there are no singular points for the equation, the solution is convergent on the entire real axis i.e. the radius of convergence is ∞.*

Example

Consider the differential equation $y'' + y' + \frac{1}{(1+x^2)} y = 0$ with Maclaurin series solution $y = \sum_{n=0}^{\infty} a_n x^n$.

- Recall distance in complex plane:

$$\begin{aligned} |z - z_0| &= \sqrt{(x - x_0)^2 + (y - y_0)^2}\ ,\ z = x + iy\ ,\ z_0 = x_0 + iy_0\ , \\ &= \sqrt{(z - z_0)\,\overline{(z - z_0)}}. \end{aligned}$$

- From the DE, we have $p(x) = 1$, $q(x) = \frac{1}{1+x^2}$, both rational.
- Complex plane: $q(z) = \frac{1}{1+z^2}$. Denominator has complex zeroes: $1 + z^2 = 0 \implies z = \pm i$, that define the two singular points for this problem.

Example continued

- Centre of series is always on real axis: $(x_0, 0)$.
- Maclaurin series: $x_0 = 0$. Hence we need distance from $(0, 0) \in \mathbb{C}$ to $\pm i = (0, \pm 1) \in \mathbb{C}$.
- Distance from $(0, 0)$ to $(0, \pm 1)$ is $\sqrt{(0-0)^2 + (\pm 1 - 0)^2} = 1$.
- Hence, the radius of convergence of the series solution is at least 1.

Another example

Determine a lower bound for the radius of convergence of series solutions of $\left(1+x^2\right)y''(x)+x^2y'(x)+xy=0$ about centre $x_0=\frac{1}{2}$.

Solution:

- Write DE as $y''+\frac{x^2}{(1+x^2)}y'+\frac{x}{(1+x^2)}y=0$.
- Denominator of both p and q is $1+x^2$.
- Write in complex plane as $1+z^2$, complex zeroes $z=\pm i$.
- Distance from centre $\left(\frac{1}{2},0\right)$ to $\pm i=(0,\pm 1)$ is $\sqrt{\left(\frac{1}{2}-0\right)^2+(0-(\pm 1))^2}=\sqrt{\frac{1}{4}+1}=\sqrt{\frac{5}{4}}$.
- Solution $y=\sum_{n=0}^{\infty}a_n\left(x-\frac{1}{2}\right)^n$ will converge with radius at least $\sqrt{\frac{5}{4}}$ about $x_0=\frac{1}{2}$.
- Interval of convergence will contain the interval $\left(\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right)$.

One more example

What is the radius of convergence of series solutions $y = \sum_{n=0}^{\infty} a_n x^n$ of the differential equation $y'' + xy' + e^x y = 0$?

Solution:

- Both coefficients are analytic functions and the equation has no singular points. Therefore, radius of convergence of series solutions is ∞.
- Thus series solutions converge for all $x \in \mathbb{R}$. The power of these techniques is that we were able to determine this without having found series solutions.

Example continues

- Find the first five nonzero terms of the specific solution in terms of a power series expansion about $x_0 = 0$ for IVP:

$$y'' + xy' + e^x y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

- The solution is an analytic function: $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$.
- Subbing in the equation:

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} a_n x^n = 0.$$

Example continues

- How to deal with

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{n=0}^{\infty} a_n x^n = 0?$$

- The easiest way to multiply the two series is to write all series in an expanded form:

$$(2a_2 + 6a_3 x + 12a_4 x^2 + \dots) + (a_1 x + 2a_2 x^2 + \dots) +$$
$$(1 + x + \tfrac{x^2}{2} + \dots)(a_0 + a_1 x + a_2 x^2 + \dots) = 0.$$

- Set the coefficients of likely powers in the left hand side to zero (these are the coefficients of the power series for 0 that is in the right hand side):
  - Coefficient of x^0:

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2}.$$

  - ... ...

Example continues

- Equating coefficients in

$$(2a_2 + 6a_3x + 12a_4x^2 + \dots) + (a_1x + 2a_2x^2 + \dots) +$$
$$(1 + x + \tfrac{x^2}{2} + \dots)(a_0 + a_1x + a_2x^2 + \dots) = 0.$$

  - Coefficient of x^0:

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2}.$$

    Coefficient of x:

$$6a_3 + 2a_1 + a_0 = 0 \Rightarrow a_3 = -\frac{a_0 + 2a_1}{6}.$$

  - Coefficient of x^2:

$$12a_4 + 3a_2 + a_1 + \frac{a_0}{2} = 0 \Rightarrow a_4 = -\frac{a_0 + 2a_1 + 6a_2}{24} = \frac{a_0 - a_1}{12}.$$

- General solution:

$$y(x) = a_0(1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \dots) + a_1(x - \frac{x^3}{3} - \frac{x^4}{12} + \dots).$$

Example continues

- We have obtained the general solution

$$y(x) = a_0(1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \dots) + a_1(x - \frac{x^3}{3} - \frac{x^4}{12} + \dots).$$

- The initial conditions $y(0) = 1, y'(0) = -1$ imply that $a_0 = 1, a_1 = -1$.
- Then the specific solution to IVP is

$$y(x) = (1 - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + \dots) - (x - \frac{x^3}{3} - \frac{x^4}{12} + \dots)$$

$$= 1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{6} + \dots$$

which provides the first five nonzero terms.

How about singular points?

- Method of Frobenius for dealing with regular singular points of a differential equation (ODE)
- Read Section 8.6 of the textbook for this method if you are interested
- I skip this part due to the time constraint, so Frobenius method won't be included in homework or final exam
- Last two lectures will be used for the final exam review session